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AN EXPERIMENTAL STUDY OF THE ANTICLASTIC BENDING OF RECTANGULAR BARS OF DIFFERENT CROSS-SECTIONS

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ABSTRACT. This paper describes a method for the survey of the surface of a beam bent by couples, with special reference to the study of the curvatures in and perpendicular to the plane of bending.

WITHIN certain limits, a uniform bar under the action of pure couples is bent into the arc of a circle, and the cross-section of the bar, originally rectangular, is now bounded by the arcs and radii of a system of concentric circles. The circumstances in which these conditions hold are determined by the condition that if the original cross-section be that of a rectangle of breadth $2b$ and depth $2c$, then b^2 must be small compared with Rc, where R is the radius of curvature in the plane of bending*. In these circumstances the transverse radius of curvature will be constant, and equal to R/σ, where σ is Poisson's ratio. A large number of methods for the determination of Poisson's ratio have been based on this relation, and one object of this paper is to provide material for a critical study of such methods and for an examination of the reliability of the value of σ so determined.

The chief points which we propose to discuss are:

(i) Is the bar, when bent by couples applied over knife-edges in the usual manner, always bent into the arc of a circle when the amount of bending is restricted to that contemplated in the theory as usually developed? If not, what curve best represents the outline of the bar?

(ii) What is the nature of the transverse curvature in similar circumstances?

(iii) How does the transverse curvature vary with distance from an origin taken midway between the knife-edges?

(iv) How does the value of Σ, the ratio of the curvatures at this origin, measured for a small amount of bending, vary with variations of the ratio of the breadth to the depth of the beam? In particular, is any departure from the value of Poisson's

* Prescott, *Applied Elasticity*, p. 40.

ratio σ, as determined directly by experiments made on the specimens concerned by means of extensometers, conditioned by the failure of the relation just given, that b^2 must be small compared with Rc?

(v) Is the value of Σ for any one bar dependent on the amount of bending? If so, what equation best represents the relation between Σ and R? In particular, is it possible to find a limiting value (Σ_0) of Σ corresponding to zero bending, which, under proper conditions, shall be in sensible agreement with σ as determined directly by the extensometer?

§ 1. HISTORICAL NOTES

F. Neumann, by fixing mirrors on the sides of a rectangular beam bent by couples, is said to have shown* the nature of the change of shape of the cross-section, but his experiments were, apparently, qualitative. Mallock† inserted four long fine steel wires into holes drilled normally into the upper surface of the beam, and determined Σ from observation of the changes in the inclination of these wires with changes in the bending couple.

If the four wires of this experiment are replaced by short pillars and mirrors are attached to the upper ends of these pillars we have a more sensitive means of measuring the change of inclination and hence determining Σ. This method has been worked out by Carrington‡.

Searle§ describes a simple method for measuring the transverse curvature by soldering two vertical rods to the sides of the bar under experiment, and measuring directly the amount of approach of their upper extremities as the beam is bent. The sensitiveness of this method has been considerably increased by I. Williams‖, who uses a mirror depending from a bifilar suspension attached to the upper ends of the rods. All these are open to criticism, inasmuch as the beam is tampered with by solderings or by other methods of attachment made in the immediate neighbourhood of those parts of the beam whose behaviour is under investigation. Moreover, in most of these experiments no attempt is made to correct for initial curvature, and the values of Σ so found are subject to considerable variations among themselves.

Optical methods have been used, and these, in so far as they involve no tampering with the bar, must be considered as superior to the methods already described. Cornu¶ first pointed out that the fringes produced in monochromatic light by the air film between the bent beam and a flat surface form a family of hyperbolas, the angle α between their asymptotes being related to the ratio of the curvatures at the origin by the equation

$$\tan^2 \tfrac{1}{2}\alpha = \Sigma.$$

The method has been studied in considerable detail by Straubel** and later by Jessop††, corrections being introduced for initial curvature of the beam. In some

* *Vide* I. Williams, *Phil. Mag.* **24**, 886 (1912). † A. Mallock, *Phil. Mag.* **29**, 157 (1879).
‡ H. Carrington, *Phil. Mag.* **41**, 206 (1921). § G. F. C. Searle, *Experimental Elasticity*, p. 114.
‖ *Loc. cit.* ¶ A. Cornu, *Comptes Rendus*, **69**, 333 (1869).
** R. Straubel, *Ann. der Physik*, **68**, 369 (1899). †† H. T. Jessop, *Phil. Mag.* **42**, 555 (1921).

of the earlier experiments the distance between the knife-edges, considered in relation to the dimensions of the beam, was far too small, and Jessop has investigated the effect of the separation of the knife-edges and of the mode of support of the beam thereon. The effect of astigmatism in a beam of light reflected from the surface has been utilised by Searle* to determine the elastic constants of glass, and Andrews† has shown that Σ may be determined from measurements of the diffraction haloes produced when lycopodium powder is dusted on the polished surface of a bent beam. Baker‡ uses interference methods, but solders the holder for the interferometer on to the bar. He finds that Σ is independent of the stress up to values of the stress of 10,000 lb./sq. inch, and that for rectangular bars of different cross-section, Σ is a linear function of the ratio of the breadth to the depth, decreasing as this ratio increases.

§ 2. THEORY

The theory of the anticlastic bending of a rectangular bar under the action of couples is fully discussed by Love § and Prescott ||. It is sufficient for our purposes to note that if, taking axes as shown in Fig. 1 (*f*), we assume a system for which the stress component X_x is equal to Eaz ¶, all the remaining stress components vanishing, we find without difficulty that the equation to the curve into which points originally on the x axis are displaced is

$$z = -\tfrac{1}{2}ax^2,$$

and that the bending moment over any cross-section is $EaAk^2$. Hence the beam is bent into a parabola, and the radius of curvature R at any point of the beam is given by

$$R^{-1} = a - \tfrac{3}{2}a^3x^2,$$

approximately. Thus as far as elementary theory is concerned the identification of the curve of bending with a circle is justified if x^2/R^2 is negligibly small in comparison with unity. The shape of the distorted cross-section is readily worked out, and it is seen that the lines

$$z = \pm c$$

become

$$z = \pm c + \tfrac{1}{2}a\sigma(y^2 - c^2),$$

which is simply the parabola

$$z = \tfrac{1}{2}\sigma a y^2$$

displaced. Hence both longitudinal and transverse curvatures are similar in character, and σ is simply the ratio of the radii of curvature of these parabolas *at the origin*. It follows that a method such as that devised by Mallock, which measures curvatures in the neighbourhood of the origin, is, *ceteris paribus*, more

* G. F. C. Searle, *Proc. Cam. Phil. Soc.* 21, p. 772 (1923).
† Jas. P. Andrews, *Phil. Mag.* 2, 945 (1926).
‡ Baker, *Phil. Mag.* (1924).
§ Love, *Mathematical Theory of Elasticity.*
|| Prescott, *Applied Elasticity.*
¶ E stands for Young's Modulus.

likely to give results in accordance with elementary theory than those methods which depend on the assumption of a circular outline over a wide range.

The relations which we have been discussing hold for points between the knife-edges, and indeed, in terms of the general theory, they hold there only for those points for which x^2/R^2 is small in comparison with unity. Outside the knife-edges we are dealing with a modified cantilever problem, and the equation to the curve into which points originally on the x axis are displaced is of the type

$$y = A - Bx + Cx^2 - Dx^3.$$

It will, however, be seen in the sequel that, empirically at least, the curve of longitudinal section of a bar bent by couples can be very closely represented by equations which are valid both inside and outside the knife-edges, holding good almost to the points of application of the load.

§3. METHODS

(*a*) The longitudinal curvature of a bar bent by couples may conveniently be studied by coating the edge of a bar with white paint and photographing the painted edge. Lantern slides prepared from these photographs, and projected on to a screen of squared paper, enable the x, y co-ordinates of the curve with respect to any arbitrarily chosen co-ordinate axes to be read off at once. The lenses used were tested for any possible distortions by photographing a *réseau*, a straight-edge and an arc of a circle, and measuring the resultant photographs. The results of these tests were quite satisfactory. Some attempts were made to study the transverse curvature by laying a straight-edge across the bent bar, and photographing the gap between the straight-edge and the bent bar. The resulting negative was measured under the microscope, but the result obtained could only be considered a rough approximation.

(*b*) An optical method was used which had the advantage that it permitted the contouring of the whole surface of the bent beam, were that considered desirable. A very fine pencil of light, isolated from a parallel beam by means of a stop with a pinhole aperture, impinges on the bent plate at its centre of symmetry, so that it is reflected along its incident path, and is received on a screen distant about 2 metres from the bar, where it forms a small but definite patch of light. The bar is then traversed by equal steps along a line perpendicular to the normal at its centre, and the position of the reflected spot is observed. From these observations the co-ordinates of any point on the line which originally coincided with the x axis may easily be deduced.

Let RQP represent (on an exaggerated scale) the required curve, and let $RS = QT = d$, the amount of each horizontal traverse. With the notation shown it may be seen that $\angle PQT = \frac{1}{2}\phi_1$ and $\angle QRS = \frac{1}{2}(\phi_1 + \phi_2)$, so that if $PT = y_1$, $P'R = PT + QS = y_2$, then

$$y_1 = d \tan \tfrac{1}{2}\phi_1,$$

$$y_2 = d\,[\tan \tfrac{1}{2}\phi_1 + \tan \tfrac{1}{2}(\phi_1 + \phi_2)],$$

and in general

$$y_n = d\,[\tan \tfrac{1}{2}\phi_1 + \tan \tfrac{1}{2}(\phi_1 + \phi_2) + \ldots + \tan \tfrac{1}{2}(\phi_{n-1} + \phi_n)].$$

If the deflections are such that we may replace tangents by angles,

$$y_n = d\,(\phi_1 + \phi_2 + \ldots + \phi_{n-1} + \tfrac{1}{2}\phi_n).$$

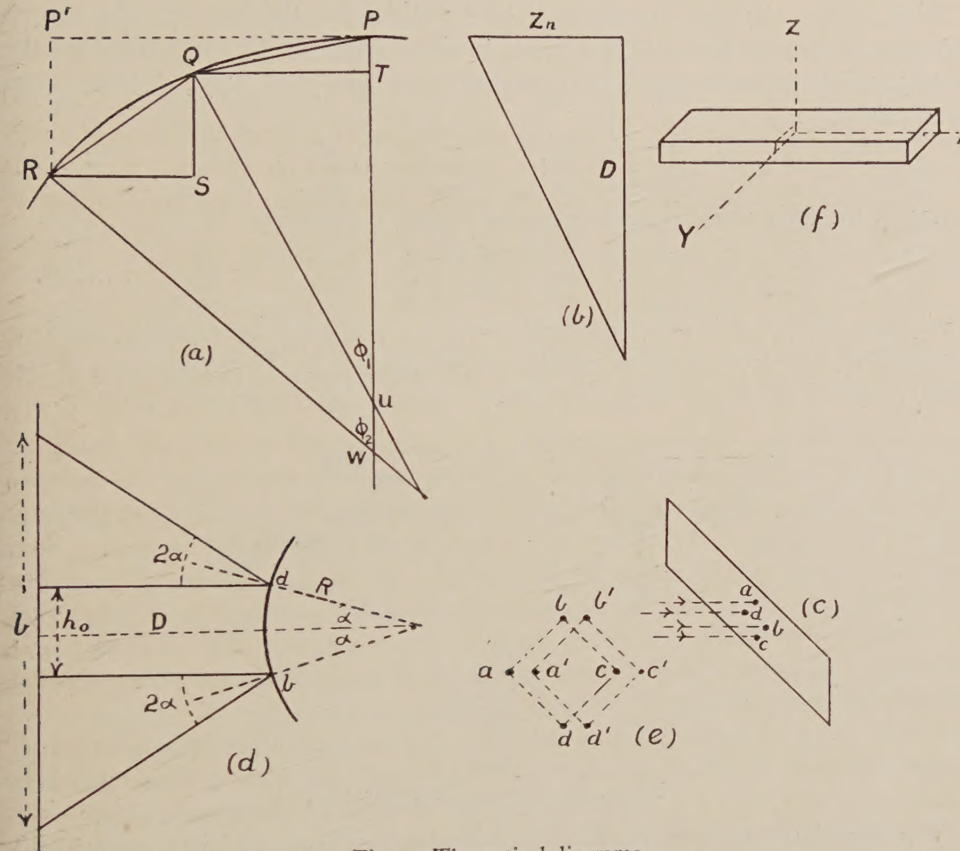


Fig. 1. Theoretical diagrams.

Suppose now that the reflected beam falls on a screen placed at a shortest distance D from the plate, Fig. 1, so that the beam reflected from P meets it normally. If then, corresponding to a traverse of x_n (equal to nd), the reflected beam is turned through an angle $2\phi_n$, and the spot on the screen is deviated through a distance z_n, we have

$$\phi_n = z_n/2D,$$

and

$$y_n = \frac{d}{2D}\left(z_1 + z_2 + z_3 + \ldots + z_{n-1} + \frac{z_n}{2}\right) \qquad \ldots\ldots(1).$$

The required curve is now obtained by plotting values of y_n against the corresponding values of x_n. In a similar manner the transverse curvature may be investigated.

The ratio Σ of the curvatures in the neighbourhood of the origin may or may not coincide with the value of Poisson's ratio σ as determined directly by

experiments with the extensometer. Σ is most readily determined by using pencils of light in place of the thin wires of Mallock's experiment. Four fine pencils are isolated from a parallel beam of light by means of a stop pierced with four pin-holes placed approximately at the corners of a square. These four pencils fall on the bar as shown at (*c*) and are reflected as before on to a screen at a distance D.

If h_0 and v_0 be the "horizontal" and "vertical" distances on the screen corresponding to the points (*d* and *b*) and (*a* and *c*) respectively, and h and v are the corresponding distances when the plate is bent, we have, Fig. 1 (*d*), with the usual assumptions,

$$\frac{h-h_0}{2D}=\frac{bd}{R}; \qquad \frac{v-v_0}{2D}=-\frac{ac}{R_1}.$$

Hence Σ, the ratio of the curvatures at the origin, is given by

$$\Sigma=-\frac{bd}{ac}\frac{v-v_0}{h-h_0} \qquad \text{......(2).}$$

If we, therefore, plot $(v-v_0)$ against $(h-h_0)$ we shall, if Σ be constant, obtain a straight line, the slope of which will be proportional to Σ. If Σ vary, it is, at any point, proportional to the ratio of ordinate to abscissa at the point in question.

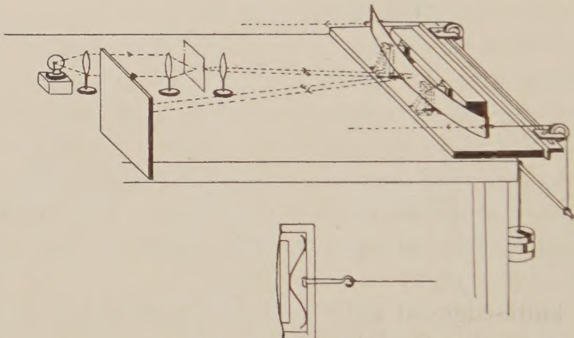


Fig. 2. Experimental arrangements.

The experimental arrangements are indicated in Fig. 2, which is almost self-explanatory. The source of light was a Pointolite, and the manner of applying a variable couple to the bar is shown in sufficient detail in the figure. By traversing the bar in the manner already explained, a series of values of z could be obtained, and from these the corresponding values of y deduced by means of equation (1). A series of readings was always taken on the unbent plate, so that a correction might be applied if necessary to the values obtained in any particular experiment.

§4. RESULTS

(i) *The longitudinal curvature of a uniform rectangular bar bent by couples.*

A brass bar of length 60 cm., breadth 5 cm., depth 0·23 cm., was supported against two knife-edges 20 cm. apart in the manner shown in Fig. 2. Table I below shows the results of an exploration of the bar by the method of the deflected pencil.

Table I: Typical results for brass bar.

$D = 239$ cm., $d = 2{\cdot}54$ cm.; hence $d/2D = 0{\cdot}00533$.

	Bent by couple 2×10^4 gm. cm.		Correction, from unbent plate	Correct	Bent by couple 4×10^4 gm. cm.		Correct
x_n	z_n	y_n		y_n	z_n	y_n	y_n
cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.
0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
2·5	1·35	0·0036	0·0055	0·0091	4·50	0·012	0·017
5·1	3·50	0·016	0·021	0·037	9·15	0·048	0·069
7·6	6·20	0·042	0·042	0·084	14·20	0·110	0·152
10·2	9·75	0·085	0·063	0·148	20·65	0·202	0·265
12·7	10·70	0·138	0·090	0·228	23·55	0·320	0·410
15·2	13·10	0·202	0·124	0·326	28·40	0·459	0·583
17·8	21·80	0·294	0·142	0·436	39·45	0·635	0·777
20·3	26·00	0·421	0·138	0·559	45·25	0·865	1·00
22·8	25·50	0·560	0·137	0·697	45·95	1·10	1·24
25·4	24·10	0·692	0·146	0·838	45·35	1·35	1·50
27·9	24·80	0·821	0·163	0·984	44·75	1·58	1·74

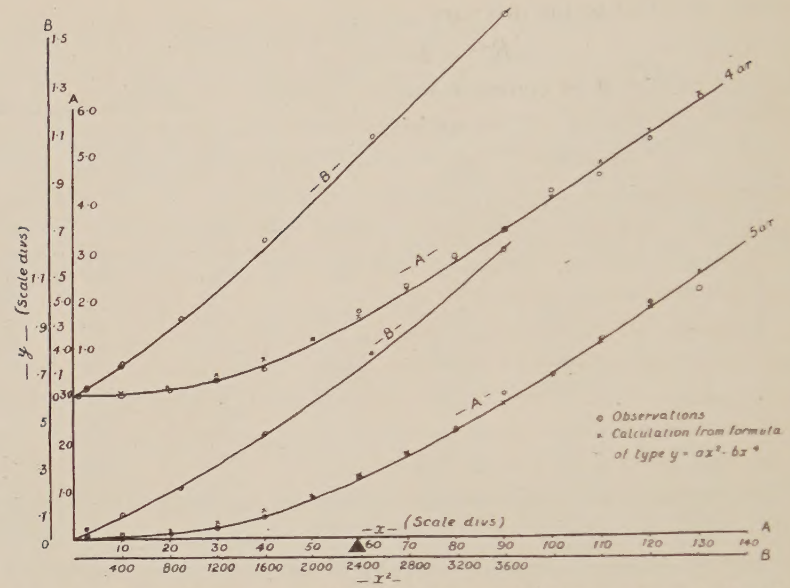
These figures are typical of a large number obtained. As we have seen, elementary theory assumes a parabolic outline, which for small strains becomes indistinguishable from a circle. That is, when y is very small compared with x it is a matter of indifference whether we plot x^2 or $(x^2 + y^2)$ against y to obtain a rectilinear graph. In actual fact, a plot of x^2 against y shows distinct curvature (see Fig. 3) for the portion of the beam between the knife-edges, even though the elevation of the beam at the centre is quite small. This serves to emphasise the statement that measurements of longitudinal curvature depending on applications of the formula $R = (x^2 + y^2)/2y$ to a beam bent by couples (y being the elevation midway between knife-edges at a distance of $2x$ apart) must be made with considerable caution, and this fact should be borne in mind in measurements of Young's modulus made by uniform bending methods.

If we plot y/x^2 against x^2 we obtain a curve which is very fairly rectilinear from the origin almost to the points of application of the load. There is no marked change in passing through the points at which the knife-edges are situated, and we may therefore take the equation

$$y = ax^2 - bx^4 \qquad \text{......(3)}$$

as representing with considerable accuracy the outline of the beam over almost its entire length*. Fig. 3 shows the outline of two bars, and gives some conception

* In some of the specimens examined the equation $y = ax^2 - bx^4$ strictly represented the facts throughout the whole range considered; in other specimens the simple parabolic formula represented the facts over the region between the origin and the first knife-edge, with perhaps a little greater accuracy. The fourth power term represents over this region only a very small addition, and its introduction displaced the calculated points from the experimental points by amounts which were only a very little greater than the unavoidable errors of experiment; but since the displacements were in every instance in the same direction, we have deemed it advisable to record this fact.



Curves A, profiles; Curves B, y against x^2 for the portion between inner knife-edges.

Fig. 3. Profiles of longitudinal sections of bars.

of the range over which they may be represented by a formula of the type of equation (3). Table II below shows in detail the closeness with which this equation represents the experimental facts.

Table II

Dimensions of bar, 60 × 5 × 0·23 cm. Formula, $y = ax^2 - bx^4$.

x_n	Couple 2 × 10⁴ gm. cm.		Couple 4 × 10⁴ gm. cm.	
	$a = 1{\cdot}464 \times 10^{-3}$, $b = 2{\cdot}494 \times 10^{-7}$		$a = 2{\cdot}65 \times 10^{-3}$, $b = 5{\cdot}3 \times 10^{-7}$	
	y_n observed	y_n calculated	y_n observed	y_n calculated
cm.	cm.	cm.	cm.	cm.
0·0	0·0	0·0	0·0	0·0
2·5	0·009	0·0091	0·017	0·0164
5·1	0·037	0·0379	0·069	0·0684
7·6	0·084	0·0838	0·152	0·1514
10·2	0·148	0·1494	0·265	0·2698
12·7	0·228	0·229	0·410	0·4129
15·2	0·326	0·325	0·583	0·5837
17·8	0·436	0·439	0·777	0·787
20·3	0·559	0·561	1·003	1·001
22·8	0·697	0·694	1·237	1·235
25·4	0·838	0·841	1·496	1·488
27·9	0·984	0·989	1·74	1·741

Now if the longitudinal outline is parabolic, and given, say, by

$$y = ax^2,$$

the curvature, as far as terms in x^2 are concerned, is given by

$$R^{-1} = 2a - 12a^3x^2,$$

and it is usual to take it as constant and equal to $2a$. If we take equation (2) as representing the outline of the bar we easily find that

$$1 + (2ax - 4bx^3)^2 = (2a - 12bx^2)^{\frac{2}{3}} R^{\frac{2}{3}}.$$

The additional terms are often quite appreciable for the values of x met with in the ordinary cases, where measurements are made on the assumption of circular curvature, and it is not uncommon to find that the curvature at the knife-edges may differ from the curvature at the origin by as much as 1 or 2 per cent.

It may be noted that the curves connecting x^2 and y flatten out as the bending is increased, and that at considerable curvatures—in many instances quite outside the range suggested by the elementary theory—the longitudinal curvature becomes very closely parabolic. This condition may be very closely realised in the thinner plates which may be subjected to considerable bending without serious permanent deformation. An instance is given in Table III below.

Table III

Rolled brass bar, 30·6 × 2·80 × 0·062 cm. Couple = 1600 gm. cm.

x	0	10	20	30	40	50	60	70	80	90	100	110	120
$x^2 . 10^{-3}$	0	1·0	0·4	0·9	1·6	2·5	3·6	4·9	6·4	8·1	10·0	12·1	14·4
y, cor.	0	0·2	0·8	1·8	3·4	5·2	7·8	10·8	14·1	18·1	22·3	27·1	31·7
x^2/y	—	0·50	0·50	0·50	0·47	0·48	0·46	0·46	0·46	0·45	0·45	0·44	0·45

Similar investigations were carried out by the alternative photographic method described, with results which confirm those arrived at by the deflected beam method. An example of the results obtained is shown in Table IV below.

Table IV

Rolled brass bar, 30·3 × 2·78 × 0·137 cm. Formula, $y = ax^2 - bx^4$.

x_n	Couple 8000 gm. cm.		Couple 17,000 gm. cm.	
	$a = 0·000352$, $b = 2·21 \times 10^{-9}$		$a = 0·000665$, $b = 5·00 \times 10^{-9}$	
	y_n observed	y_n calculated	y_n observed	y_n calculated
cm.	cm.	cm.	cm.	cm.
0·0	0·0	0·0	0·0	0·0
1·17	0·0	0·03	0·1	0·07
2·34	0·1	0·14	0·2	0·26
3·51	0·2	0·31	0·4	0·70
4·68	0·4	0·56	0·8	1·05
5·85	0·8	0·87	1·4	1·64
7·02	1·2	1·24	2·15	2·32
8·19	1·7	1·67	3·1	3·14
9·36	2·2	2·17	4·0	4·06
10·53	2·9	2·70	5·1	5·05
11·70	3·4	3·30	6·2	6·15
12·87	4·1	3·94	7·3	7·30
14·04	4·7	4·61	8·5	8·54
15·15	5·0	5·33	9·9	9·78

(ii) *The transverse curvature of a uniform rectangular beam bent by couples, and*
(iii) *The variation of transverse curvature with distance along the beam.*

This curvature was investigated by the reflected beam method, with the difference that it was found more convenient in traversing any given section to move the perforated diaphragm rather than to move the bar. The curvature was investigated at the origin and at different points along the bar. Table V shows a typical series of figures.

Table V

Dimensions of bar, 60 × 5 × 0·23 cm.

Transverse distance from centre y_n	Mean deflection z_n		z_n correct	Mean deflection z_n		z_n correct
	Bent	Unbent		Bent	Unbent	
cm.	cm.	cm.	cm.	cm.	cm.	cm.
0·0	0·0	0·0	0·0	0·0	0·0	0·0
0·5	0·18	1·52	1·02	1·30	1·62	0·32
1·0	1·12	3·14	2·02	2·48	3·27	0·79
1·5	1·95	4·92	2·97	3·90	4·87	0·97
2·0	3·90	6·95	3·05	5·60	7·15	1·55
	$x = 0$			$x = 18$ cm.		

If we assume that the transverse curvature is constant and equal to R_1^{-1}, we may test this constancy rapidly as follows. We have, with symbols analogous to those in Fig. 1,

$$2\phi_n = z_n/D = 2y_n/R_1,$$

$$R_1 = 2D.\frac{y_n}{z_n} = 400y_n/z_n,$$

since D was equal to 2 metres.

Hence, if we plot y_n against z_n, a straight line will be obtained if the curvature is uniform, and this curvature will be numerically equal to 1/400 of the slope of the line. In every instance examined the lines so drawn were rectilinear within the limits of experimental error, so that the transverse curvature may be assumed uniform at all points along the bar. An investigation of the variation of transverse curvature with distance along the bar yielded the results shown in Table VI.

A plot of R against x^2 was strictly linear over a wide range, both inside and outside the knife-edges. In the instance given in Table VI the results were very closely represented by

$$R_1^{-1} = 4{\cdot}12 \times 10^{-3} - 6{\cdot}4 \times 10^{-6}x^2 \qquad \text{......(4)},$$

as is shown by the third column in Table VI, and the curve of Fig. 4. It will be noticed that there is no marked alteration in the curve after its passage through the position occupied by the knife-edge, but that equation (4) represents the experimental facts up to a point near the point of application of the load. It is evident that, if this equation holds, there will be a point for which the transverse

curvature vanishes, and as a matter of fact such a point does exist, although, as Fig. 4 shows, its position is not given by equation (4). The point was further investigated by observation of the interference fringes formed as in experiments made by Cornu's method. Photographs were taken of the fringes on a bent glass bar, and these photographs show quite clearly that the curvature vanishes, and indeed changes sign at a point close to the end of the bar (see Plate I).

Table VI

Distance from centre	Transverse curve	
	Observed	Calculated
cm.	cm.	cm.
0	$43 \cdot 8 \times 10^{-4}$	$41 \cdot 2 \times 10^{-4}$
3	38·8	40·6
6	34·8	38·9
9	35·2	36·2
12	32·2	32·0
15	26·2	26·8
18	18·2	20·5
21	11·2	13·0
24	7·0	4·4
27	5·3	− 5·4

(iv) *Variation of the ratio of the principal curvatures at different points along a rectangular bar bent by a constant couple.*

It follows that, if the longitudinal curvature of a beam bent by a definite fixed couple is given by

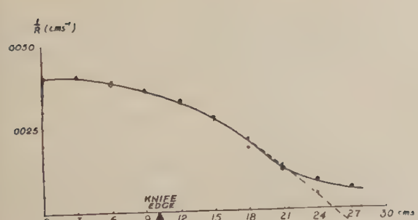
$$R^{-1} = A - Bx^2 + Cx^4,$$

and the transverse curvature is given by

$$R_1^{-1} = A_1 - B_1x^2,$$

then Σ, the ratio of these curvatures at any given point, will in general vary as we move along the bar, and will be given by

$$\begin{aligned}\Sigma &= R_1^{-1}/R^{-1}\\ &= (A_1 - B_1x^2)(A - Bx^2 + Cx^4)^{-1}.\end{aligned}$$



o = Observations.
$\times$ = Points on curve, equation (4).

Fig. 4. Variation of transverse curvature with distance from centre.

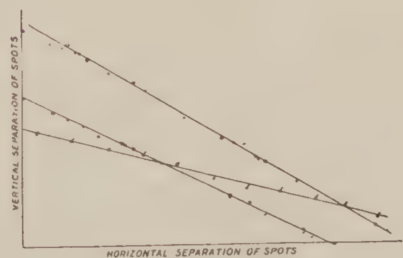


Fig. 5. Observations made by four-spot method.

(v) *The determination of Σ, the ratio of the principal curvatures at the origin; its variation with variation in the thickness of a rectangular bar of constant breadth; and its relations to σ, the value of Poisson's ratio, as determined by extensometer experiments.*

The method employed for the determination of Σ was the four-spot method described previously, see Fig. 1 (*c*) and (*d*). The apparatus used was that sketched in Fig. 2, with the substitution of a four-orifice for a single-orifice stop. Σ was calculated from equation (2), and the amount of bending was small. In this series of experiments, as indeed in all the experiments described in this paper, the knife-edges were always separated by more than three times the width of the bar.

In any one series of experiments on one bar, readings were taken with the bar bent alternately convex and concave, the bending was then increased, and the observation repeated. A curve was plotted between v and h, and the value of Σ was deduced from the slope of the curve. With the amount of bending employed these curves were always rectilinear (see Fig. 5).

For each bar this series of operations was repeated from four to six times, and a mean value of Σ taken.

The bar under experiment was then tested in an Avery 10-ton testing machine, which was placed at our disposal by the kindness of Professor E. H. Lamb. Longitudinal extensions and the corresponding lateral contractions were both read by instruments of Professor Lamb's* design. Fig. 6 (Plate I) shows a bar under test, with the extensometers *in situ*. Plots of the deflections of the two extensometers for different bars are exhibited in Fig. 7, and it will be seen that they are strictly rectilinear, so that Poisson's ratio can be deduced at once from the slopes of these curves.

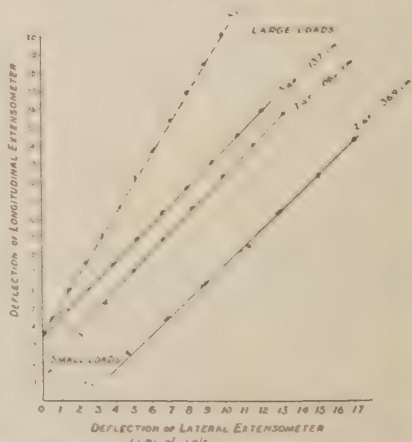


Fig. 7. Examples of testing machine results. (Slopes are inversely proportional to Poisson's ratio.)

It was not possible to test all the bars in this way, as some of the thinnest buckled under the pressure of the lateral extensometer.

Experiments were made with two specimens of brass. From the first specimen a number of strips of length about 30 cm. were cut. These strips were each about 2·9 cm. wide and originally 0·32 cm. thick. They were cut down to different thicknesses in the workshop of the East London College. Each specimen was polished on one side by means of a succession of graded emeries, and finished to uniform overall thickness. Table VII following shows the values of Σ, the ratio of the curvatures at the origin for a series of bars about 30 cm. long, 2·9 cm. wide, and of a thickness varying from about one-third of a centimetre down to one-third of a millimetre. They are compared with the values of Poisson's ratio σ as determined directly by means of the testing machine.

* E. H. Lamb, *Engineering*, Feb. 13, 1925.

The plates from the second specimen, instead of being cut down to different thicknesses, were rolled by Messrs Johnson & Matthey. They were annealed, and finally polished and finished to even thickness overall at the East London College. Table VIII shows the results of tests made on these specimens.

Table VII.

Thickness	Σ	Mean Σ	σ
cm.			
0·315	0·32		
	0·31		
	0·33		
	0·31		
	{0·32}		
	{0·30}	0·315	0·323
0·221	0·30		
	0·26		
	0·23		
	0·24		
	0·28	0·262	0·245
0·144	0·28		
	0·28		
	0·27		
	0·26	0·273	0·290
0·114	0·25		
	0·26		
	{0·23}		
	{0·26}		
	0·23	0·246	0·292
0·084	0·35		
	0·32		
	0·36		
	0·36		
	0·36		
	0·36	0·351	0·423*
0·062	0·49		
	0·57		
	0·52		
	0·56	0·53	0·342
0·038	0·76		
	0·66		
	0·63†		
	0·70	0·69	

* Verified subsequently.

† Too thin for testing machine.

Table VIII.

Thickness	Σ	Mean Σ	σ
cm.			
0·466	0·36		
	0·33		
	0·32		
	0·36	0·343	0·323
0·369	0·30		
	0·33		
	0·32		
	0·31	0·315	0·354
0·262	0·32		
	0·32		
	0·32		
	0·32	0·320	0·354
0·173	0·30		
	0·31		
	0·30		
	0·30	0·303	0·321
0·137	0·28		
	0·33		
	0·29		
	0·29		
	0·33	0·30	0·336
0·084	0·33		
	0·32		
	0·33		
	0·36	0·335	0·355
0·062	0·39		
	0·37		
	0·37		
	0·38	0·378	0·323
0·038	0·61		
	0·63		
	0·61		
	0·57	0·60	0·32
0·026‡		(0·76)‡	

‡ Too thin. Value given is rough approximation only.

Two possible sources of error were investigated. The plates were not polished to an optical surface, and it is probable that the variations in Σ shown in a set of experiments on any one plate may be due in some measure to accidental irregularities in the surface. This was tested by an experiment in which a stop with eight orifices was used in order to give two different sets of readings from which to compute Σ. The eight orifices were so arranged as to form two squares contiguous each to the other (cf. Fig. 1 (*e*)). In Table VII above, bracketed values of

Σ indicate numbers which were obtained in this way. It may be mentioned, as confirming this view, that values of Σ obtained from experiments on a glass bar with a good optical surface show less variation *inter se*. The effect of additional annealing was also investigated, and Table IX below shows the effect on the value of Σ produced by re-annealing certain plates.

Table IX

Plate	7 ar.	8 ar.	9 ar.
Thickness, cm.	0·062	0·038	0·026
Σ after first annealing	0·378	0·60	0·76
Σ after re-annealing and re-polishing	0·384	0·448	0·82

It will be seen that though there is a definite effect due to additional annealing there is no alteration in the trend of the variation of Σ with thickness. The general result of the two sets of experiments is exhibited in Fig. 8. It is evident that in both sets of experiments Poisson's ratio σ as directly measured is not subject to

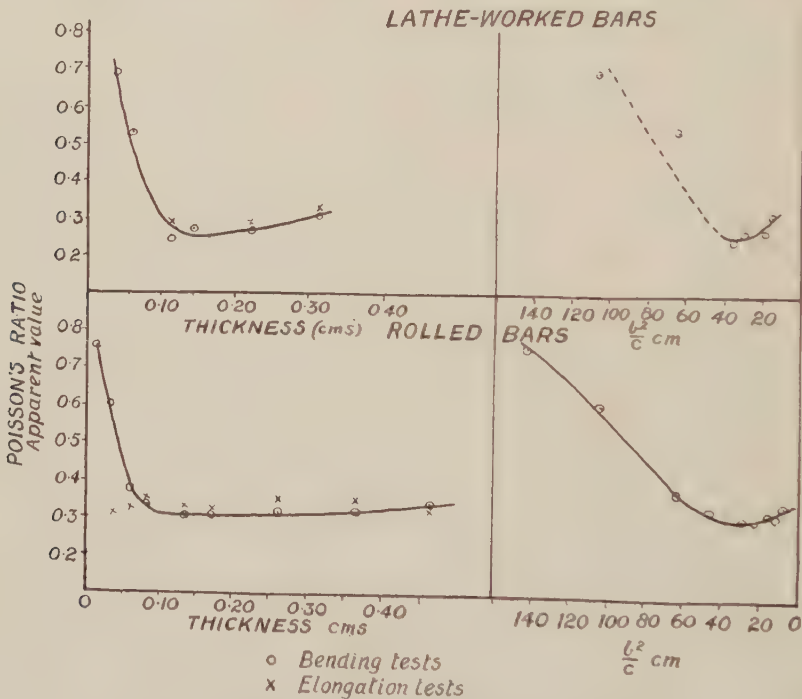


Fig. 8. Poisson's ratio for rolled and lathe-worked bars.

large variations over the range of thickness considered. The quantity Σ, defined as the *ratio at the origin of the longitudinal to the transverse radius of curvature*, decreases slightly with decreasing thickness and then rapidly increases. The variation with b^2/c is shown in the same figure, and as an average value of the radius R

of longitudinal curvature employed in these experiments was about 290 cm., it will be seen that the rapid increase of Σ begins at values of b^2/Rc of the order 0·14.

Table X: Constants in formula $y = axe^{-bx}$.

Plate	Thickness	a	b	Remarks
	cm.			
7 ar.	0·062	0·489	0·00828	Rolled strips
8 ar.	0·038	0·891	0·0152	Rolled strips
8 b.	0·084	0·339	0·00643	Strip cut at E.L.C.

Finally, we proceed to consider how the value of Σ for any one bar varies with the longitudinal curvature R. The experiments were carried out in the manner described in discussing the series immediately preceding, the longitudinal curvature being, however, pushed to values far exceeding those contemplated in the quantitative development of elastic theory. If $h - h_0$ be denoted by x, and $v - v_0$ by y, so that, as previously explained, Σ is proportional to y/x, it is found that, over the range of experiment, a plot of $\log (y/x)$ against x yielded a straight line. Hence, the experimental results are expressed by the equation

$$y = axe^{-bx},$$

values of a and b for some of the plates tested are shown in Table X above, and the agreement between calculated and observed values of x and y is shown in Fig. 9.

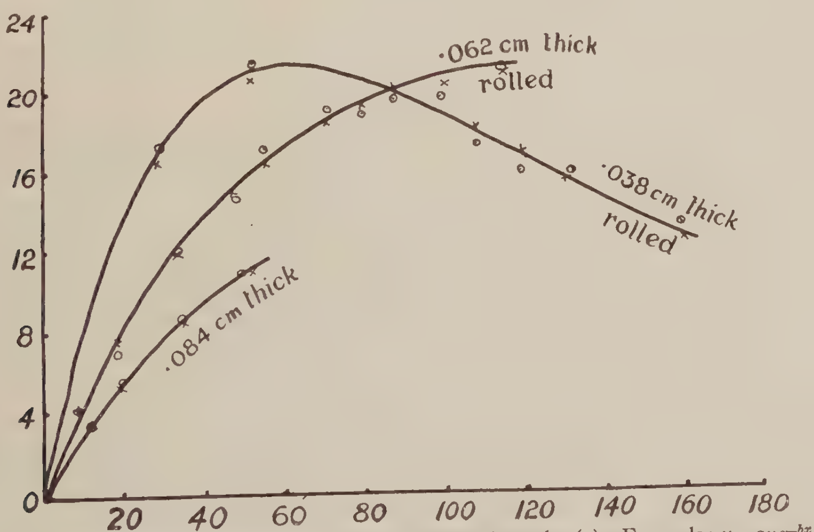


Fig. 9. Comparison of calculated (×) and observed results (o). Formula: $y = axe^{-bx}$.

Since y/x is proportional to Σ, and x is proportional to the longitudinal curvature R^{-1}, it follows that we may put

$$\Sigma = \Sigma_0 e^{-k/R}.$$

Table XI below gives values of Σ_0 and k, calculated from this formula for certain of the plates previously tested. The values of Σ_0 should be compared with the

values of Σ obtained for small finite bendings and exhibited in Tables VII and VIII. The formula, which does not profess to do more than express the result of experiments made over a definite range of curvatures, shows no indication of a possible change of sign in Σ.

Table XI: Constants in formula $\Sigma = \Sigma_0 e^{-k/R}$.

Plate	Thickness	Σ_0	k	Remarks
	cm.			
7 ar.	0·062	0·476	2·73	Rolled strips
8 ar.	0·038	0·891	5·01	Rolled strips
8 b.	0·084	0·339	1·86	Strip cut at E.L.C.

To sum up: We are now in a position to answer some of the questions propounded at the outset.

(i) The shape of the bar, from origin almost to the point of application of the load, is accurately represented by the equation

$$y = ax^2 - bx^4,$$

and the corresponding longitudinal curvature by the relation

$$1 + (2ax - 4bx^3)^2 = (2a - 12bx^2)^{\frac{2}{3}} R^{\frac{2}{3}}.$$

(ii) The transverse curvature is, within the limits of experimental error, constant.

(iii) The transverse curvature at different points along the bar is given by

$$R_1^{-1} = A_1 - B_1 x^2.$$

(iv) The value of Σ, the ratio of the curvatures at the origin, decreases slightly with increase in the ratio b^2/c, and then increases rapidly. If Σ is to be identified with Poisson's ratio σ, such identification must be confined to the region where $b^2/Rc < 0{\cdot}14$. In this region, comparison with extensometer values justifies the identification of Σ with σ.

(v) For any one bar, the value of Σ over a considerable range of bending is connected with the longitudinal curvature by the equation

$$\Sigma = \Sigma_0 e^{-k/R}.$$

(vi) Under the heading (iv) we have pointed out that within certain limits, experimentally determined, Σ may be taken as a measure of Poisson's ratio. Of the various methods proposed for the measurement of Σ, those are to be avoided which involve soldered attachments to the bar under test. Optical methods are, therefore, to be preferred, and of these methods, the authors find the deflected beam method employed in the present research simple and reliable in practice, and of the order of sensitiveness of ordinary interference methods.

Our thanks are due, and are tendered, to Professor C. H. Lees and to Professor E. H. Lamb for the facilities which they have placed at our disposal.

DISCUSSION

Mr T. Smith: I would suggest that a hyperbolic or somewhat similar form for the bent bar might be anticipated from very general considerations when the positions of the knife edges, the points of application of the forces, and the dimensions of the bar can be left out of account. The form found would then be that for a bar of length great compared with the distances between the points mentioned; at considerable distances from these points it would be natural to suppose the bar distorted but little from its original shape. The formula found by the authors for the central longitudinal section $x = ay^2 - by^4$ is not inconsistent with the hyperbolic equation $\left(\frac{n+\alpha}{\alpha}\right)^2 - \frac{y^2}{\beta^2} = 1$ for small values of x. The curvature of the transverse section would be a maximum at the centre and diminish to zero at a large distance. It would be interesting to know whether the authors' observations are consistent with the bar's assuming the form of a slice of a hyperboloid of revolution, with the axis of rotation parallel to the axis of the unbent bar. If this form were correct the ratio of the axes of the central section would be equal to Poisson's ratio, and the form of the bent bar would be completely specified by a single variable.

Mr A. G. Warren: It seems possible that the reduction of the transverse curvature at the knife edges may be due to the fact that the knife-edge reaction tends to keep the bar flat. The differences in the mean values found for Poisson's ratio and for the ratio of the curvatures at the origin might perhaps be due to lack of homogeneity in the material, in consequence of which different methods of experiment would give different results.

Mr Andrews (in reply to Mr Smith): A very large number of expressions have been tried, and those given in the paper best represent the shape of the bent surface. While we have not gone into the matter from Mr Smith's point of view, we do not anticipate that the surface could be represented as he suggests.

I am in agreement with Mr Warren as to the cause of the reduction of curvature, but that explanation of the results was purposely omitted for the sake of brevity. It is very improbable that lack of homogeneity is at the root of the variation of the results for bars of different thickness; all rolled plates were annealed twice or three times, yet they gave the same type of variation as the lathe-worked bars, which were either unannealed or annealed once. Moreover, it is unlikely that lathe-working would have produced the same kind of heterogeneity as rolling.

AN INSTRUMENT FOR THE PRODUCTION OF KNOWN SMALL HIGH FREQUENCY ALTERNATING ELECTROMOTIVE FORCES

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ABSTRACT. A portable instrument for the production of known electromotive forces, variable in frequency from 10 to 50 kilocycles and in magnitude from 0·0076 to 15,000 microvolts, is described. The instrument is intended for the calibration of amplifiers and the measurement of the strength of wireless signals of long wave-length.

There are many electrical measurements which necessitate the production of a small electromotive force of radio frequency and known magnitude. For example, a wireless wave produces in a receiving aerial high frequency electromotive forces which are too small for direct measurement, and high and low frequency amplification, with rectification, must usually be employed before a deflection can be obtained on a measuring instrument. If a small electromotive force of known variable frequency and magnitude is injected into the receiving aerial, equal deflections of a measuring instrument will be produced, when the electromotive force induced in the aerial by the wireless wave is equal in frequency and magnitude to the locally produced electromotive force. In order to use this method of measuring the strength of wireless signals, electromotive forces of radio frequency whose magnitudes are known accurately must be produced. Portable apparatus for this purpose has been described by several writers[(1)] and a summary of published work to September, 1926, is available[(2)].

The portable instrument shown in Fig. 1 and Fig. 5, Plate I, is designed to cover the frequency range 10 to 50 kilocycles and the intensity range 0·0076 to 15,000 microvolts. Alternating current, variable in frequency over the desired range, is generated by a valve oscillator enclosed in metal screens which prevent the formation of electric and magnetic fields external to the screens. A coil coupled inductively to the valve oscillator and connected in series with an A.C. milliammeter supplies a known alternating current to the input of an artificial line composed of series and shunt resistance elements, described in detail later. Each shunt element of this line carries $1/\sqrt{2}$ times the current carried by the preceding element, so that the magnitude of the electromotive force available at the signalling key at the left of the diagram depends upon the section of the artificial line to which a rotating switch makes connection. By depressing this signalling key, the desired electromotive force may be injected into the amplifier circuit at a suitable place.

§ 1. THE SCREENED VALVE OSCILLATOR

The circuit used with the instrument consists of a coil tuned by a condenser and a six-valve high and low frequency amplifier with heterodyne, as shown in Fig. 1. It is necessary to screen the valve oscillator until the stray electric and magnetic fields from it are too feeble to induce a detectable electromotive force in this receiving circuit at a distance of 6 ft. As the circuit is sensitive to an electromotive force of 0·01 microvolt induced in the coil, adequate screening of the valve

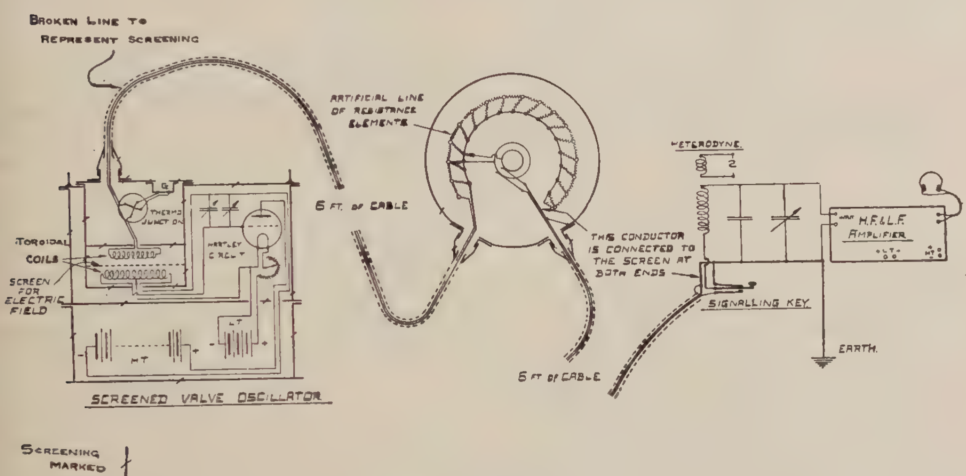

Fig. 1. Signal strength meter: screening and simplified connections.

oscillator is difficult. In the final form of instrument the magnetic and electric fields in the neighbourhood of the valve oscillator are reduced to a safe limit in three ways. The valve oscillator generates just sufficient high frequency electrical energy to produce a suitable deflection of the thermal milliammeter in the secondary circuit. The coil in the oscillatory circuit of the valve oscillator is wound in toroidal form to reduce its external magnetic field, while the external electric and magnetic fields remaining after taking these precautions are confined by enclosing the valve oscillator in metal screens.

A dull emitter valve with a 66-volt high tension battery and three dry cells for filament heating generates sufficient high frequency electrical energy. Continuous frequency variation from 10 to 50 kilocycles is obtained with a number of fixed condensers controlled by a rotary switch, a variable condenser and a toroidal coil of 50 millihenries inductance. The circuit oscillates freely and supplies practically constant high frequency current to the secondary circuit over the whole range of frequency. The current in the secondary circuit is adjusted to the desired values by altering the valve filament current with a rheostat. The fixed coil of the oscillatory circuit is wound on a dust core, $3\frac{3}{32}$ in. internal diameter, $5\frac{3}{32}$ in. external diameter and 1 in. thick with 800 turns of No. 36 S.W.G. double-silk-covered copper

wire in a single layer. A coil of this type, wound as shown on Fig. 2 (*a*), produces a much weaker external magnetic field than a coil of similar size and inductance wound with circular turns in the usual way. In spite of this a quantity of energy sufficient to induce a detectable electromotive force in the receiving circuit penetrated the screens. By a slight modification in the winding, shown in Fig. 2 (*b*), the external magnetic field was almost eliminated and the little that remained was reduced by the screens to such an extent as to be negligible.

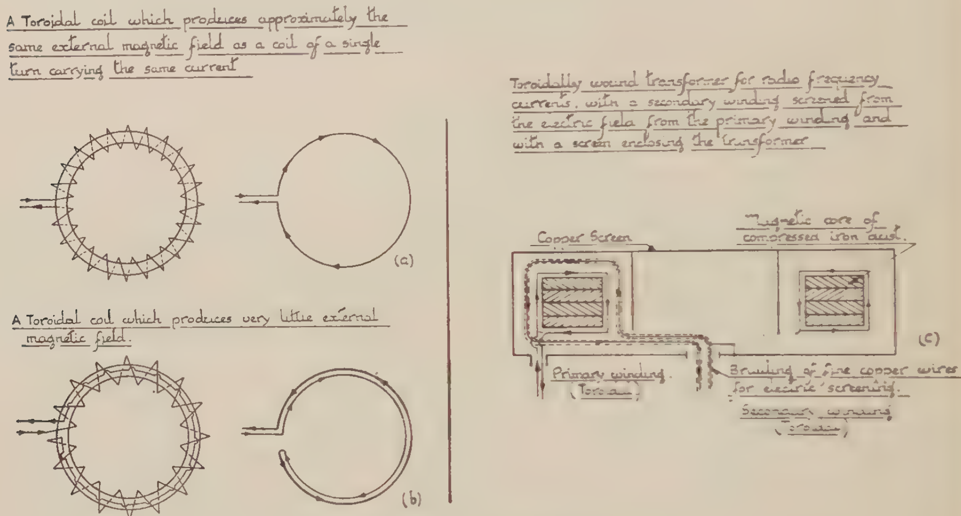

Fig. 2. Screened transformer.

Two secondary windings of 4 and 12 turns respectively are wound spirally and symmetrically round the toroid so as to reduce their external magnetic field as much as possible. The secondary windings are screened from the electric field of the primary winding by a braiding of fine copper wires, as shown in Fig. 2 (*c*), the braiding being connected to the metal screen round the coils. Either of these secondary windings may be connected through a vacuum thermo junction of suitable range to the artificial line by a rotating switch. The wires in this part of the circuit are twisted together in pairs to reduce stray magnetic fields.

Stray magnetic fields from the valve oscillator were more difficult to eliminate than stray electric fields. The coil which produces practically all the stray magnetic field is placed in a soldered copper box, less efficient screening being inadequate (see Fig. 3 and Fig. 7, Plate I). The vacuum thermo junctions and the switch controlling them are sufficiently screened by a copper box with a copper lid fastened with screws. The screening of the remainder of the circuit which produced most of the stray electric field was comparatively simple, two boxes and a lid cast in a light alloy of nickel chromium and aluminium being satisfactory. The dry batteries are placed in a separate compartment, so that any corrosive liquid leaking from a defective cell can do little harm.

Dials and switches external to the screens, marked *A*, *B*, *C* and *D* in Fig. 3, are provided for controlling the thermo-junction switch, the fixed air condensers, the variable air condenser and the filament rheostat respectively. These dials can be handled without producing any detectable stray fields. The control of the variable air condenser is effected by a metal rod, rotating in a metal bush in the screen and making electrical contact therewith. This rod carries, outside the screen, an ebonite handle, and inside the screen is attached to the shaft carrying the moving vanes of

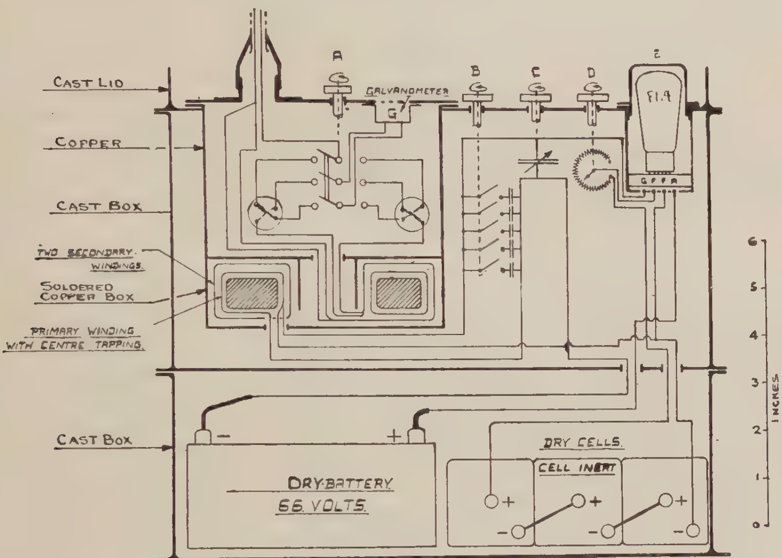

Fig. 3. Screened valve oscillator.

the condenser by a coupling of insulating material. The remainder of the condenser is fixed to the inside of the screen, but insulated from it so that the condenser is completely insulated from the screen. The change in the resistance of the screen produced by a metal rod passing through it and making good electrical contact is insufficient to cause a detectable change in the efficiency of the screening. The other external controls are similarly designed.

§ 2. THE ARTIFICIAL LINE

The artificial line consists of a network of resistances in 35 sections, each section comprising a series element of resistance r and a shunt element of conductivity g, as shown in Fig. 4. Alternating current from the valve oscillator enters the line at 1 and is divided at 2, a small part of it returning to the valve oscillator through a shunt element and the remainder passing to 3. The process is repeated at each section of the line so that each shunt element carries less current than the preceding element in a constant ratio. A calculation from the usual equations for a resistance network shows that, with a series resistance of 0·0347 ohm and a shunt conductance g of 3·47 ohms, this ratio is $1/\sqrt{2}$. The voltage across each shunt element is also $1/\sqrt{2}$

times that across the preceding element, so that a rotation of the switch making contact with the ends of a shunt element by one stud in a clockwise direction reduces the potential difference between the leads to the key in the ratio $1/\sqrt{2}$. The input resistance of the line is 0·12 ohm, so that the electromotive force available at the signalling key is 1000 microvolts when the alternating current entering the line is 8·33 milliamperes and the rotating switch makes contact at 1. The electromotive forces available with the switch making contact at other sections of the line form the geometrical series 1000, 707, 500, 353, ... 0·0076, the common ratio of which is $1/\sqrt{2}$. By supplying a current of 125 milliamperes to the line,

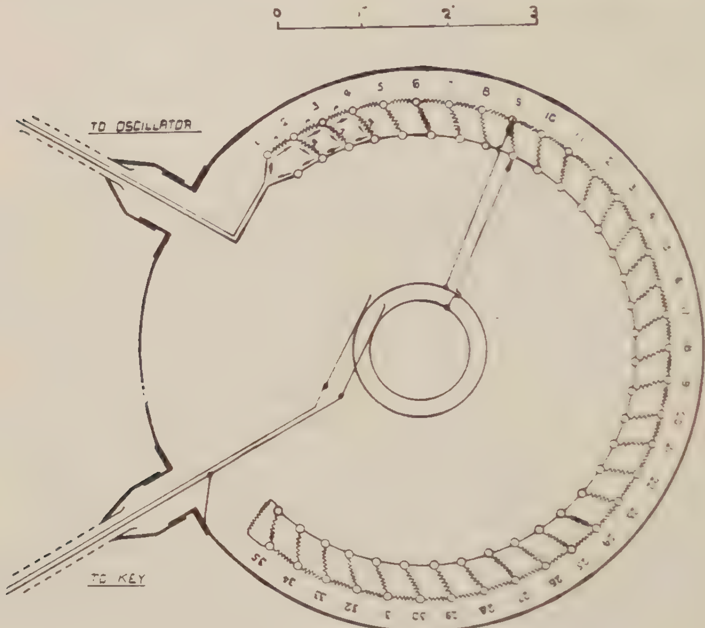

Fig. 4. Artificial line.

another geometrical series of electromotive forces ranging from 15,000 to 0·114 microvolts can be obtained. The current entering the line is measured with a suitable milliammeter connected in circuit with the switch *A* in Fig. 3.

The accuracy of the line at the highest frequency of 50 kilocycles per second depends upon the self-inductances of the elements and the mutual inductances between them. An inductance of 0·011 microhenry has a reactance of 0·00347 ohm at this frequency, which is 10 per cent. of the resistance of a series element. By arranging the connections so that loops of wire are avoided, the inductances of the elements of the line can be reduced to less than this amount. The attenuation factor of a line composed of resistance elements with comparatively small inductances in series with them can be calculated from the standard equations involving hyperbolic functions for artificial lines. It appears that the effect of the small inductances is mainly to introduce a progressive change of phase along the line. The change in the

ratio of the magnitude of the currents in successive elements is only a second order effect and is negligible provided that the reactance of an element is less than 10 per cent. of its resistance. Fortunately the change of phase along the line is of no importance in the present application.

These conclusions are confirmed by the following measurements of the ratio of the voltage across one shunt element to that across the preceding element, using direct current and alternating current at high frequencies.

Table: Ratios of voltages in adjacent shunt elements.

Sections	Direct current	Alternating current (kilocycles per sec.)			
		10·1	45·1	185	500
1–2	0·711	0·712	0·711	0·703	0·670
2–3	0·718	0·710	0·716	0·708	0·679
3–4	0·709	0·711	0·708	0·701	0·674
4–5	0·707	0·707	0·708	0·703	0·671
5–6	0·707	0·711	0·703	0·706	0·681
Mean	0·710	0·710	0·709	0·704	0·675

The attenuation factor is practically independent of frequency up to 185 kilocycles. Above this frequency a calibration must be made, a method described by D. W. Dye(3) being very convenient. By redesigning the artificial line to reduce still more the self and mutual inductances, its calibration may be made independent of frequency, at least up to 500 kilocycles. A further extension of the frequency range can be achieved by increasing the resistance of the series and shunt elements in the same proportion, thereby reducing the importance of the residual self and mutual inductances. For use at much higher frequencies, an artificial line in which both series and shunt elements are inductances would be better. The calibration of such a line should be independent of frequency over a very wide range, the lower limit being determined by the resistances of the elements and the upper limit by the capacities between them.

The wiring of the artificial line is shown in Fig. 6, Plate I, in which the series elements (No. 22 S.W.G. Eureka wire) can be seen connected to the outer row of studs. The shunt elements (No. 28 S.W.G. Eureka wire) bridge across the two rows of studs. The terminal resistance of 0·12 ohm which makes the line of 35 sections equivalent to an infinite line can be seen to the left.

§ 3. THE SIGNALLING KEY

It was found that any method of injecting an electromotive force into the resonant circuit, shown in Fig. 1, involving an abrupt change in the constants of the circuit, such as a momentary increase in its resistance, produced a noise in the head telephones. A signalling key was designed which produced no electrical disturbance in the amplifier and its circuits and no noise in the telephones. In the resting position of the key, shown in Fig. 1, the middle contact presses against the upper contact, the artificial line being disconnected. A partial depression of the

key brings the middle contact into connection with the bottom contact, connection with the top contact being maintained. Complete depression of the key disconnects the top contact, the electromotive force from the artificial line being then in series with the circuit. A resistance of about 0·1 ohm which is added to the circuit at the same time is usually too small to produce an appreciable effect.

§ 4. THE RESIDUAL ELECTRIC AND MAGNETIC FIELDS EXTERNAL TO THE SCREENS

The continuous metal screen, completely enclosing the valve oscillator, the artificial line and the signalling key, is by no means a sufficiently perfect conductor for complete screening. A large number of joints, most of them only held together by screws, introduce resistances. The currents induced in the imperfect screen by the magnetic and electric fields generated inside it thus differ somewhat in distribution, magnitude and phase from those induced in a perfectly conducting screen. These induced currents, opposed by the resistance of the screen, give rise to potential differences over its outer surface. There are, therefore, residual electric and magnetic fields outside the screen. However, the electromotive force induced by the residual magnetic vector field from the screened valve oscillator in a coil of 10 millihenries inductance and 18 in. in diameter at a distance of 6 ft. is less than 0·01 microvolt — a quantity which can be neglected. The residual electric vector field was further reduced by connecting a point on the screen to earth. It is not possible completely to eliminate it by an earth connection because the screen is not an equipotential surface. The distribution in space of the residual electric field depends upon the point on the screen to which an earth connection is made, the point chosen being that which results in the smallest electric field in the neighbourhood of the tuned circuit and amplifier. With the connections between screen, circuit and earth, shown in Fig. 1, the residual electric field is too small to be detected, that is to say, the electromotive force induced by it in the tuned circuit connected to the amplifier is less than 0·01 microvolt. An accidental earth connection to another part of the screen alters the potential distribution over the outside of the screen and the resulting external electric field sufficiently to produce a detectable electromotive force in the tuned circuit. The effect of the high resistance connection to earth, introduced by touching any part of the screen, is not detectable.

§ 5. USE FOR THE CALIBRATION OF VALVE AMPLIFIERS

In order to form an estimate of the quality of a valve amplifier, it is desirable to know the smallest electromotive force which can be detected, the largest electromotive force which may be amplified without distortion, the amplification factor between these limits, and the frequency range over which the amplifier is satisfactory. A knowledge of the smallest electromotive force which can be detected is of considerable interest. An amplifier giving a high overall amplification accompanied by loud disturbance from Schrott effect, mechanical vibration, interference from neighbouring electrical circuits, etc., may be inferior to an amplifier having a lower

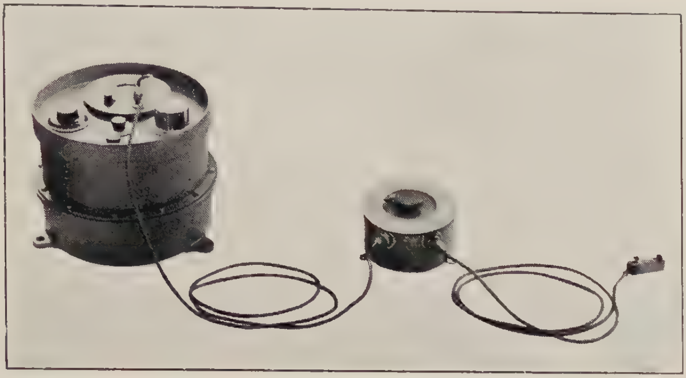

Fig. 5. The complete instrument.

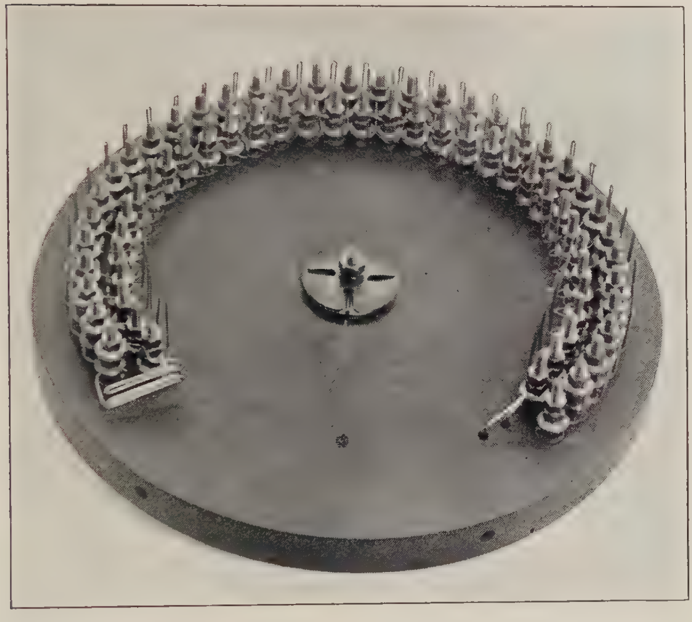

Fig. 6. Wiring of the artificial line.

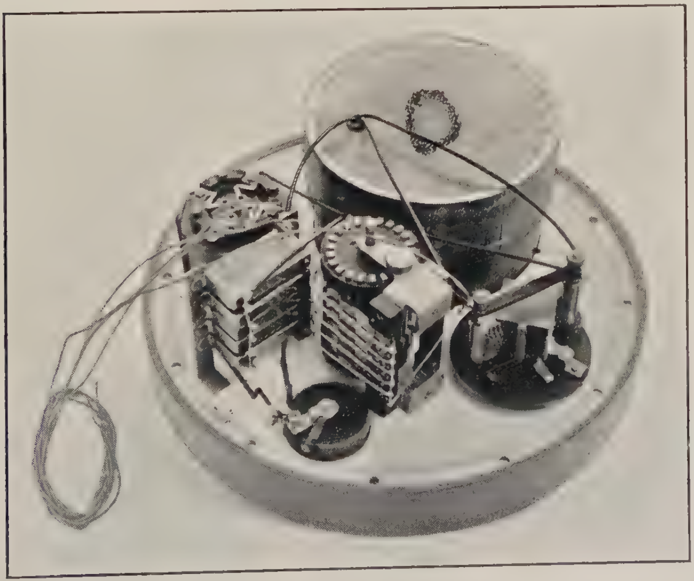

Fig. 7. Interior of valve oscillator.

overall amplification, but which is comparatively free from disturbances. A measurement of the minimum electromotive force required to produce an audible signal may be made by the arrangement of circuits shown in Fig. 1. One observer depresses the signalling key at irregular intervals and reduces the magnitude of the injected electromotive force until another observer, listening with the telephones, fails to hear the signals. The best settings for heterodyne, filament current, anode voltage, reaction, etc., are easily found by the same procedure. The overall amplification is measured by replacing the telephone with a measuring circuit of the same impedance and observing the deflections which correspond to known electromotive forces, injected into the circuit by depressing the signalling key. The contribution to the total amplification made by each stage of high frequency amplification may sometimes be measured by injecting an electromotive force in all the grid circuits in turn and observing the corresponding amplified electromotive forces.

If the electromotive force, injected into the circuit by depressing the signalling key, as shown in Fig. 1, is increased in strength, the amplified electromotive force is found to increase correspondingly until the last valve begins to produce distortion. Further increase in the injected electromotive force at first produces an increase in the amplified signal and then a decrease. Measurements of this kind are valuable in that they indicate the range in which an amplifier may be used without producing distortion. Many similar measurements with amplifiers for which the instrument is suitable will suggest themselves to those who are familiar with the technique of high frequency alternating current.

§ 6. USE FOR THE MEASUREMENT AND COMPARISON OF THE STRENGTH OF SIGNALS

The instrument is well adapted for the measurement of the strength of wireless signals of long wave-length. The receiving circuit, shown to the right of Fig. 1, becomes a receiving circuit for wireless waves if a frame aerial is substituted for the coil in the tuned circuit. On depressing the signalling key, an artificial wireless signal is produced which can be adjusted in frequency magnitude and duration until it matches the true wireless signal very closely. It is not possible to match exactly because the strength of the artificial wireless signal can only be varied in steps. In a previous section, it is shown that the electromotive forces which are available form a geometrical progression in which the ratio between successive terms is $1/\sqrt{2}$. It is therefore possible to find a known electromotive force not differing by more than 20 per cent. from any unknown electromotive force within the range 15,000 to 0·0076 microvolts. A measurement of an unknown electromotive force is made by comparing it with two known electromotive forces, one slightly stronger and the other slightly weaker. The comparison may be made by listening to the artificial and the true signals with head telephones with an error of less than 10 per cent., even when considerable interference from atmospherics, etc., is present. If a more accurate measurement is desirable, a recording or indicating instrument is substituted for the head telephones.

§ 7. CONCLUSION

Several of these instruments have been in continual use for over a year and have been found to be robust, easily portable and reliable.

The authors desire to express their indebtedness to the Admiralty for granting permission to publish this paper and their appreciation of the assistance given by Mr H. Bellars and Mr L. O. Cook, of the Admiralty Research Laboratory, in the mechanical design and construction of the instrument.

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DISCUSSION

Mr R. S. Whipple: I should like to congratulate the authors on the very practical apparatus they have designed. It is an extremely difficult matter to achieve such effective screening.

Dr D. Owen: The authors offer a solution of an important practical problem in high-frequency testing. The all-important point, however, as to the adequacy of their artificial line to supply voltages according to a simple exponential law is not treated theoretically in the paper. It seems desirable that this basic assumption should be justified, say, by reference to some publication containing a full examination of the point. The steps taken, with such success, to avoid leakage of magnetic and electric fields are of much interest. The method of eliminating stray field due to the toroid is notable in that it recognises the fact—usually quite overlooked—that the current in the winding will not only produce circular magnetic lines within its interior, but also generate a field outside, in virtue of its acting as a one-turn coil coincident with a central line within the toroid. This is, of course, true of every solenoidal helical winding, which is thus not the complete equivalent of the mathematical current sheet of theory.

Dr W. H. Eccles: Workers in my laboratory in the early days of amplifiers found it necessary to take extreme precautions in screening the oscillator used for supplying the current to be amplified by the instrument under test. Complete enclosure in metal did not seem good enough, so the amplifier was placed in another room and the leads coming from the oscillator were carried through "compo" piping. It was found necessary to close up all gaps in the piping. Even then stray disturbance direct from the oscillator was not all eliminated. Instead of an artificial line I used a calibrated variable mutual inductance and a potentiometer in which the resistance was a liquid electrolyte. I experienced difficulty from the fact that tapping the potentiometer disturbs the voltage distribution in the potentiometer circuits, especially when the impedance of the tapping circuit was low. Have the authors considered this difficulty in their use of an artificial line?

Mr R. M. Wilmotte (communicated): The authors are to be congratulated on the very ingenious method they have devised to obtain a range of E.M.F. whose extremes are in the high ratio of $2 \cdot 10^6$ to 1. An artificial line for the purpose is far more satisfactory than a potentiometer for, while the latter's scale is linear, the former's is logarithmic. There is, moreover, the considerable advantage that the input impedance of the line is extremely low and capacity effects correspondingly negligible. The greatest of these capacity effects is, no doubt, the input impedance of the amplifier. The authors suggest replacing the resistance elements by inductances for higher frequencies. I should think that the design of such a line with the elimination of mutual inductances would be extremely difficult, especially as the decrement of these inductances as well as their impedances would have to be low (if capacity effects are to be neglected). The difficulty with both resistances and inductances is that, owing to impurities, they vary with frequency. The purest form of impedance available is a condenser. I should be very interested to hear the authors' views on the possibility of using large condensers instead of either resistances or inductances for high frequencies. Except at extremely high frequencies, the inductances could be made negligible while capacity-to-screen effects could readily be taken into account in the design, even though the impedance of the line were large. The minimum size of condenser that could be used would be dependent on the input impedance of the amplifier. It might be necessary to bring the line on to the lower part of the resonance curve in order to obtain a measurable current in the thermo-ammeter, but there would be no great objection to this. If this produced too large voltages, the artificial line could be connected to a capacity potentiometer (two condensers in series, the line being connected across the larger one). As a small point of interest, I would like to know why a screen for the electric field is used between the primary and secondary of the toroid.

Authors' reply: The description which Dr W. H. Eccles has given of the difficulties which he encountered in measuring the sensibility of an amplifier and the methods he adopted to overcome them are of great interest. His experimental screened cable in which two insulated wires were run through a lead or "compo"

pipe could hardly be improved upon, its only disadvantage being its limited flexibility. The liquid electrolyte as a non-inductive potentiometer would be very satisfactory provided that the circuit into which it was introduced had a sufficiently high resistance. The artificial line described in the paper was wound to a low resistance of about 0·1 ohm so that only a circuit of exceptionally low resistance would disturb it.

We are unable to give Dr D. Owen a reference to a publication in which the theory of the artificial line as a means of producing small known electromotive forces is specifically presented. However, it is a special case of the electrical filter circuit, the generalized theory of which has been given in numerous publications, references to some of which follow:

(1) Campbell, *Bell System Technical Journal* (Nov. 1922). (2) Zobel, *ibid.* (Jan. 1923). (3) Zobel, *ibid.* (Jan. 1925). (4) Peters, *Journ. A.I.E.E.* **42**, 445 (1923). (5) Peters, *Phil. Mag.* **6**, no. 34 (1928). (6) Peters, *ibid.* **3**, no. 16 (Supplement) (1927).

The properties of the artificial line can be inferred from first principles by the application of Kirchhoff's equations for electrical networks. If Z_1 be the complex impedance of a series element and Z_2 the complex impedance of a shunt element, the ratio, α, of the current in the $(n-1)$th shunt element to that in the nth element is given by the equation

$$1/\alpha + \alpha = 2 + Z_1/Z_2.$$

In general, α is a complex quantity from which the decrement of the artificial line and the change of phase per section can be inferred. The decrement of an artificial line whose shunt and series elements are any combination of resistance, capacity and inductance can be found by solving the above equation. The calculated decrement of the artificial line must usually be verified by a calibration against some precision method of producing small known electromotive forces. The method described by Dr D. W. Dye to which a reference has been given is very suitable for this purpose.

Mr R. M. Wilmotte has summarized the advantages of the artificial line very concisely. The range of electromotive forces which must be measured in the testing of valve amplifiers is so large that a measuring instrument with a logarithmic scale is very desirable. His objection to our suggestion of an artificial line for high frequencies, in which both series and shunt elements would be inductances, is that residual mutual inductances between adjacent sections of the line would introduce errors however carefully the line were constructed. Provided however that a few sections at the beginning and end of the line were avoided, the greater part of such a line would have a constant decrement per section. A calibration of the line would be necessary. Mr Wilmotte's alternative of a line with condensers for series and shunt elements seems quite practicable and has, as he points out, marked advantages at very high frequencies. The electric screen between the two windings of the toroidal transformer seems superfluous theoretically but is found experimentally to be essential.

THE EFFECT OF MOIST AIR ON THE RESISTANCE OF PENCIL LINES

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ABSTRACT. The resistance of a pencil line is found to increase when it is kept in a moist atmosphere. The change in resistance for different degrees of moisture is investigated and a suggestion is made that this change may, in certain circumstances, be utilised to measure humidity.

During a measurement of the resistance of a pencil (carbon) line by the Wheatstone bridge method it was observed that the presence of moisture in the air caused a gradual increase in the resistance of the line. The object of the present investigation was to study the changes in the resistance of a line when it was subjected to a series of humidity changes and to see how far this property could be utilised for measurements of humidity.

The resistance of the various lines used was of the order of 0·1 megohm. Appropriately high resistances formed the other three arms of the bridge. Instead, however, of balancing every time to find the absolute resistance of the carbon line, the null position was obtained once only at the beginning of each series of experiment and the change in the resistance of the line under the changing conditions obtained by noting the galvanometer deflections round about the null point, the equivalence between deflection and change in resistance being known by a previous calibration.

The pencil lines were drawn on rectangular pieces of ground glass or of ebonite, the surface of the latter having been roughened by means of emery paper, previous to the drawing of the lines on them. The ends of the lines were broadened out and tinfoil-covered rubber pieces were used to secure good contact at the ends. The line under experiment was kept inside a wide glass tube through which could be passed a current of dry or moist air by means of a suitable aspirator arrangement. The air was dried by passing it through calcium chloride tubes or by allowing it to bubble through concentrated sulphuric acid. The humidity of the moist air was controlled by means of suitable sulphuric acid-water solutions.

A current of air (dry or moist) was blown through the tube for a given time and changes in the resistance of the line were measured after regular intervals of time. Before the change due to a given humidity was measured the line was exposed to a current of dry air for at least half an hour and the change in its resistance determined every few minutes. Air after bubbling through a sulphuric acid-water

solution of known strength (from which the humidity of the outgoing air could be calculated) was then passed through the tube containing the line, and the changes in the resistance measured for 25 minutes. Dry air was again passed for half an hour or more before subjecting the line to the influence of moist air of a different humidity, and so on.

The resistance of the line changes also with temperature and with time. The temperature variation, while the line was subjected to one particular humidity or to a current of dry air, was never more than about a fifth of a degree centigrade; and since the temperature coefficient of the resistance was approximately 2×10^{-3} this caused a change in the given resistance of only about 1 in 2500. As regards the change of resistance with time, it was found that with a freshly prepared line the change per hour was of the order of 1 in 10,000 in the beginning. This reduced to about 1 in 20,000 after 20 days, and to 1 in 100,000 after 35 days. These changes can therefore be regarded as negligible for our present purposes.

When a current of moist air is passed over one of these lines, which has been previously exposed to a current of dry air, its resistance rises at first rapidly and then more slowly, attaining a sort of saturation value in about 25 minutes. On subjecting the line to the influence of dry air at this stage the resistance falls fairly rapidly. In the case of ground glass, it was observed that the line on being exposed to rather high humidities showed, at first, the increase of resistance, but then a decrease. The latter was found to be due to the moisture condensed on the glass, which caused a lowering of the insulation resistance of the glass itself. On coating the portion of the glass outside the line with a thin layer of wax the above mentioned anomalous behaviour disappeared. Such a pencil line (i.e. a line on a ground glass piece the surface of which, outside the line, is waxed) is, therefore, the one suitable for our purposes. With a line on an ebonite piece there is no such trouble.

For a line on ebonite the change of resistance in the first 25 minutes was nearly 1 in 50 when the absolute humidity was 42 mm. and about 1 in 250 for a humidity of 26 mm. In the case of waxed ground glass the changes are more than twice as great. At rather low humidities (below 26 mm. for ebonite and below 10 mm. for waxed ground glass) the resistance showed either a gradual fall from the very beginning or a steady value. The changes in resistance are of course different for different lines, but they are nearly the same for the same line under similar conditions.

The curves in Fig. 1 represent the change of resistance with time for different humidities when the line was on ebonite. The curves in Fig. 2 are for a line on waxed ground glass. As previously indicated the humidity portion of the curves rises rapidly at first and then more slowly. The drying curves show a fall much more abrupt than the rise in the humidity curves. The portions of the curves corresponding to drying become nearly horizontal after about 20 minutes.

The phenomenon of a slow gradual increase of resistance of pencil lines with time, and a much more rapid increase when the line is exposed to moist air, seems to be due to air and water vapour particles finding their way into the interstices of

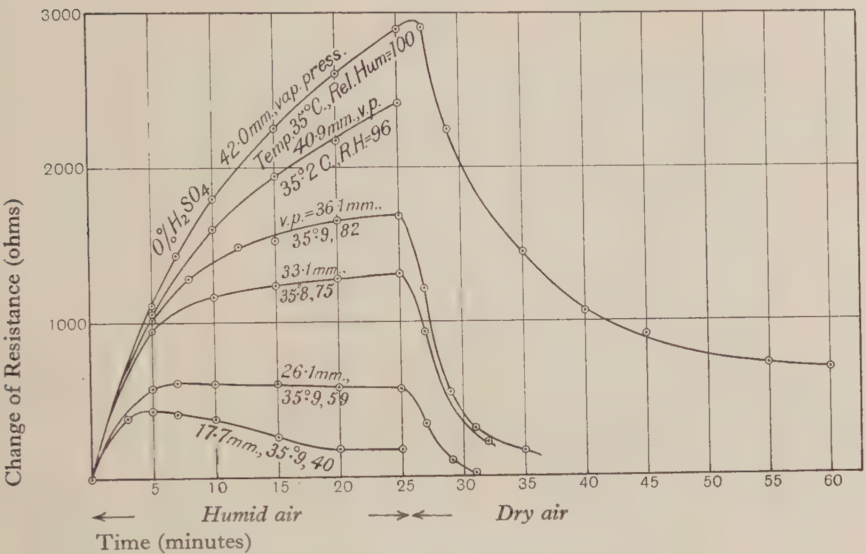

Fig. 1. Change in resistance of line on ebonite.

DATA FOR THE CURVES IN FIG. 1

Line on ebonite. Initial resistance = 155,100 *ohms*

Time (min.)		0	5	7	10	15	20	25	0 % H_2SO_4 or H_2O V.P. = 42·0 mm. T. = 35°·0 C. R.H. = 100
Change of Res. (ohms)		0	1104	1436	1796	2256	2604	2896	
Time		27	29	35	40	45	55	60	D.T. = 35°·0 C.
Change of Res.		2904	2240	1448	1064	912	728	692	
Time		0	5	10	15	20	25		V.P. = 40·9 mm. T. = 35°·2 C. R.H. = 96
Change of Res.		0	1052	1600	1940	2168	2404		
Time		0	5	8	12	15	20	25	V.P. = 36·1 mm. T. = 35°·9 C. R.H. = 82
Change of Res.		0	1012	1280	1472	1524	1654	1688	
Time		27	29	31	35				D.T. = 35°·8 C.
Change of Res.		1212	542	308	158				
Time		0	5	10	15	20	25		V.P. = 33·1 mm. T. = 35°·8 C. R.H. = 75
Change of Res.		0	952	1160	1228	1272	1309		
Time		27	30	32					D.T. = 35°·8 C.
Change of Res.		930	422	218					
Time		0	5	7	10	15	20	25	V.P. = 26·1 mm. T. = 35°·9 C. R.H. = 59
Change of Res.		0	578	608	604	600	578	570	
Time		27	29	31					D.T. = 35°·9 C.
Change of Res.		342	100	16					
Time	0	3	5	7	10	15	20	25	V.P. = 17·8 mm. T. = 35°·9 C. R.H. = 40
Change of Res.	0	396	438	416	376	264	180	174	

V.P. = vapour pressure; T. = temperature; R.H. = relative humidity; D.T. = drying temperature.

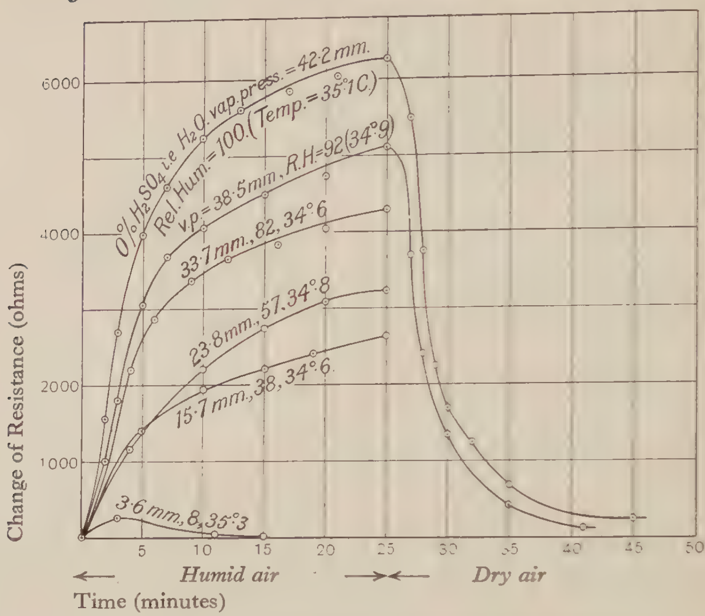

Fig. 2. Change in resistance of line on waxed ground glass.

DATA FOR THE CURVES IN FIG. 2

Line on waxed ground glass. Initial resistance = 120,000 ohms

Time (min.)	0	2	3	5	7	10	13	0 % H_2SO_4 or H_2O
Change of Res. (ohms)	0	1570	2708	3958	4610	5245	5600	V.P. = 42·2 mm.
Time	17	21	25					T. = 35°·1 C.
Change of Res.	5853	6045	6255					R.H. = 100
Time	27	28	29	30	32	35	45	D.T. = 35°·2 C.
Change of Res.	5482	3730	2258	1682	1245	685	230	
Time	0	2	3	5	7	10	15	V.P. = 38·5 mm.
Change of Res.	0	1000	1812	3062	3672	4077	4500	T. = 34°·9 C.
Time	20	25						R.H. = 92
Change of Res.	4732	5102						
Time	27	28	30	35	41			D.T. = 35°·0 C.
Change of Res.	3662	2412	1332	417	129			
Time	0	2	3	4	6	9	12	V.P. = 33·7 mm.
Change of Res.	0	1280	1855	2205	2870	3355	3653	T. = 34°·6 C.
Time	16	20	25					R.H. = 82
Change of Res.	3846	4046	4295					
Time	0	4	10	15	20	25		V.P. = 23·9 mm.
Change of Res.	0	1164	2197	2742	3072	3224		T. = 34°·8 C.
								R.H. = 57
Time	0	2	5	10	15	19	25	V.P. = 15·7 mm.
Change of Res.	0	658	1416	1956	2214	2404	2634	T. = 34°·6 C.
								R.H. = 38
Time	0	3	11	15				V.P. = 3·6 mm.
Change of Res.	0	260	35	20				T. = 35°·3 C.
								R.H. = 8

V.P. = vapour pressure; T. = temperature; R.H. = relative humidity; D.T. = drying temperature.

the line. The latter is not a continuous unbroken chain of carbon particles, but is made up of grains of carbon lying side by side, the contact between them not being perfectly rigid. The particles of air or vapour ensconce themselves in between the grains of carbon, thus increasing the resistance of the line. From the appearance of the curves in Figs. 1 and 2 (remarked at the end of the last paragraph), it would

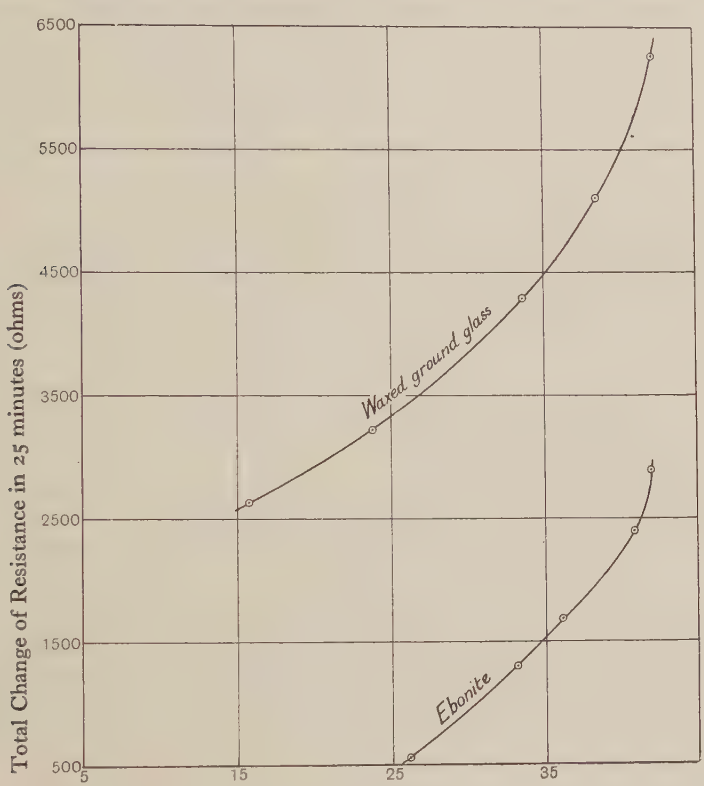

Fig. 3. Relation between change of resistance and humidity of air.

DATA FOR THE CURVES IN FIG. 3

Vapour pressure (mm.)	26·1	33·1	36·1	40·9	42·0	Ebonite
Total Change of Resistance in 25 min. (ohms)	570	1309	1688	2404	2896	
Vapour pressure (mm.)	15·7	23·8	33·7	38·5	42·2	Waxed ground glass
Total Change of Resistance in 25 min. (ohms)	2634	3224	4295	5102	6255	

appear that the rate at which vapour particles find their way into the interstices of a previously dried pencil line is slower than the rate at which they are dislodged by a current of dry air.

The curves in Fig. 3 show the relation between the absolute humidity of the air (saturation humidity being 42·0 and 42·2 mm. for ebonite and waxed ground glass lines respectively) and the changes in the resistance it causes in 25 minutes for lines on ebonite and waxed ground glass. They are fairly smooth and may be

used for determining the humidity of air. As an example, air was passed through the tube containing a line on ebonite, after first bubbling through a given sulphuric acid-water solution, and the corresponding change of resistance in 25 minutes was measured. (Dry air was passed through the tube for 30 minutes before passing the specimen air, as required by the conditions of the experiment.) The change in resistance was found to be 852 ohms and from the curve for ebonite in Fig. 3 we find that this corresponds to an absolute humidity of 29·0 mm. The density of the solution, as found by a hydrometer, was 1·27, corresponding to a concentration of about 35 per cent. acid, and a vapour pressure of 28·3 mm. The agreement between the two results is within 3 per cent., which appears to be very promising. A further study of the problem will perhaps enable us to attain a higher degree of accuracy.

At this early stage of the study of the subject it appears that the method will not be suitable when the absolute humidity of the air is less than about 26 mm. for a line on ebonite and about 10 mm. for one on waxed ground glass. The method also seems unsuitable, so far, for measuring the humidity of the atmosphere at any particular time, but can only be used to measure the mean humidity during a certain period of time. Also, since the resistance of pencil lines does not remain constant for a long period, the apparatus may require occasional calibration by passing known humidities over the line and noting the changes in its resistance.

These are some of the difficulties and limitations which would appear, at least at this early stage, to militate against the use of carbon lines for hygrometric purposes. Future experimentation on the subject may, however, overcome the difficulties. The much more abrupt fall of the drying curves as compared to the comparatively slow rise of the humidity curves might, for instance, be regarded as showing that the results obtained from the drying curves (after an exposure of the line to moist air) will probably be more useful than those from the humidity curves for hygrometric determinations.

DISCUSSION

Dr S. G. Barker, Woollen and Worsted Research Association (communicated): As one of those who are compelled to wear an electrical aid for hearing, some time ago I noted that my instrument was specially sensitive to changes in atmospheric humidity. Whereas I was able to hear perfectly whilst in a warm, dry room, on emerging into another room where the humidity was higher, I could detect a very great diminution in the efficiency of the instrument. On a rainy day it was practically impossible to use the instrument in the open air. This led me to think that there must be some intimate relation between the resistance of the carbon granules in the microphone and the humidity of the atmosphere in which the instrument was used. I therefore carried out a series of experiments in air at different humidities.

When one is slightly deaf, the ear seems to become very sensitive to even small changes in the volume of the sound, and although to the normal person this volume

seems to be hardly fluctuating at all, yet to the deaf person even these small diminutions are a matter of great moment, making all the difference between audibility and inaudibility.

It struck me that variations in humidity might explain the packing of telephone transmitters in which carbon granules are used. I found that after a time the carbon granules get covered with a dust, caused by their continually rubbing against each other, and that this dust is particularly sensitive to humidity. I have been working recently with carbon granules both in a tube and in an open trough, in order to devise a new method of humidity control and measurement, and I see from the paper that I am largely anticipated in this regard. The paper, therefore, has a greater significance than perhaps the authors recognise, in that the effect of humidity on the resistance of carbon granules is closely connected with their tendency to packing in microphones and particularly those which are used for aural aids for the deaf.

I might mention that the effect of humidity on the diminution of sound is practically instantaneous. At any rate the time lag is very small.

The efforts made so far to construct an instrument for the control or measurement of humidity have met with a fair amount of success, but are very incomplete. It is hoped that in the near future we may be able to make further progress with the instrument in our laboratories, but I felt that these remarks should be made on this paper to indicate that other workers besides the authors have been in the field.

AN ABSOLUTE CURRENT-BALANCE HAVING A SIMPLE APPROXIMATE THEORY

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AND OTHER STUDENTS OF WESTMINSTER TRAINING COLLEGE

Received September 23, 1928. *Read and discussed November* 23, 1928.

ABSTRACT. A simple form of current balance has been constructed which, when tested by experimental comparison with a standard ohm and Weston cell, measures currents with a probable error of about 1 part in 1000. The calculation of the force acting on the moving coil of the balance involves no pure mathematics beyond that of the Intermediate B.Sc. The coils are single layers so that they can in the future be made as precise helices. Actually we have had to aim at cheapness rather than at perfection, and so irregularities of shape leave the current uncertain by 5 parts in 1000. A second approximation, depending on a simple deduction from Laplace's equation, corrects the elementary theory by 1·4 in 1000 of current.

§ 1. PURPOSE

"ABSOLUTE MEASUREMENTS" are on the syllabus of the University of London for the General B.Sc. in Physics. The current balance is the most important instrument for the absolute measurement of current. But the theory of the balance at the N.P.L. involves elliptic integrals, and these are beyond the syllabus in pure mathematics for the B.Sc. aforesaid. These considerations have impelled us to make an instructional instrument. Dr F. E. Smith* has already made a simple instrument; but as our arrangement is very different, it may be worth describing.

§ 2. SIMPLE THEORY

The desired simplification of the theory has been attained by lengthening the coils in comparison with their diameter. The moving coils of Rayleigh and Mrs Sidgwick were narrow rings†. Those of the N.P.L. instrument (1907) have a length about 0·65 of their diameter. In the present instrument the length of the coils is five or six times their diameter.

The outer coils are placed in contact end to end, so that the direction of the magnetic force due to the two outer coils is that shown by the arrows in Fig. 1. The coils have a common axis, which is vertical. The inner coil hangs from the balance-beam.

Now by general theory the force which we weigh is $\partial W/\partial z$ where W is the mutual energy of the currents and z is the vertical displacement of the inner coil.

Let suffix 1 refer to the fixed coils considered as one system, while 2 is suffix for the movable coil. Let the current in all coils be j.

* F. E. Smith, Phys. & Opt. Socs. Exhibition, Jan. 1926, Catalogue B 8.

† See A. Gray, *Absolute Measurements in Electricity and Magnetism*, 1921, Ch. XII.

Then $W = j \begin{Bmatrix} \text{Number of linkages with coil 2 of tubes of induction} \\ \text{produced by coil 1} \end{Bmatrix}$ ......(1).

Hence $\dfrac{\partial W}{\partial z} = j \begin{Bmatrix} \text{Change in aforesaid linkages per centimetre axial} \\ \text{displacement of inner coil} \end{Bmatrix}$ ......(2).

As Fig. 1 shows, the tubes of induction, which enter the inner coil by both ends, escape through its central portion. Let H_1 denote the magnetic intensity due to the outer coils at a point P on the axis at the "end" of the inner coil. Then, as is well known,

$$H_1 = 2\pi n_1 j_1 \{\cos\theta_A - \cos\theta_B - \cos\theta_C + \cos\theta_D\} \quad \text{......(3)},$$

where n_1 is the number of turns per centimetre in the fixed coil, and $\theta_A, \theta_B, \theta_C, \theta_D$ are the semivertical angles of the cones subtended at P by the four "ends" of the outer coil taken in succession so that A is the near end, B and C are close together in the middle and D at the far end. See Fig. 1. The "end" of a helix is regarded as a plane normal to the axis through the mid point of the last turn.

The sizes of the coils were so chosen that, at the ends of the inner coil,

$$\partial H_1/\partial z = 0 \quad \text{......(4)},$$

which, on omitting non-vanishing factors, is equivalent to

$$0 = (\sin\theta_A)^3 - (\sin\theta_B)^3 - (\sin\theta_C)^3 + (\sin\theta_D)^3 \quad \text{......(5)}.$$

Thus the ends of the inner coil are in almost uniform fields H_1. The total number of tubes of induction entering the inner coil, of radius r_2, by its two ends is

$$2\pi r_2{}^2 H_1 = N, \text{ say,} \quad \text{......(6)},$$

and N is practically unaffected by axial displacements of a centimetre. Condition (5) means that the force does not vary while the balance swings. The antisymmetrical form of the outer current does not help in this.

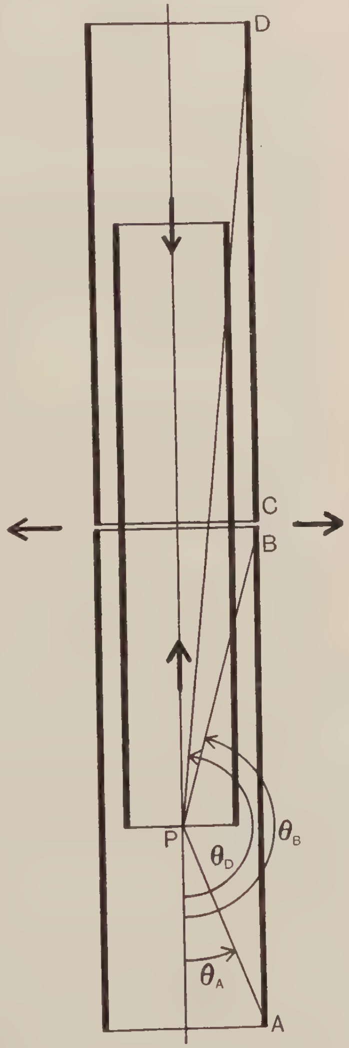

Fig. 1.

Let n_2 denote the number of turns per centimetre of axial length of the inner coil. When this coil is displaced axially by one centimetre then each of the N magnetic tubes escaping through its walls is cut by n_2 turns of wire. Hence

$$(\text{force weighed}) = \frac{\partial W}{\partial z} = jn_2 N = 2\pi j n_2 r_2{}^2 H_1 \quad \text{......(7)}.$$

Finally if $2m$ grams are needed to balance reversal of the current-connexion between the coils

$$j^2 = \frac{mg}{4\pi^2 n_1 n_2 r_2^2 \{\cos\theta_A - \cos\theta_B - \cos\theta_C + \cos\theta_D\}} \qquad \text{......(8).}$$

§ 3. CONSTRUCTION AND LENGTH MEASUREMENTS

The foregoing design was embodied as an instrument by Mr V. Stanyon, B.Sc., who solved equation (5), made coils to satisfy that condition, and with Mr G. W. Roe mounted them in position. The inner coil was wound on a bakelite tube, the outer on a tube made for the purpose by rolling up a large sheet of glued cartridge paper. The diameters were measured by a large slide caliper made by Troughton and Sims and checked for the outer coil by a steel tape (Rabone). The axial lengths were measured by a cathetometer (Pye) and a metal scale (Chesterman). In these measurements Messrs V. Stanyon, H. F. O. Dawes, C. C. Prior and W. L. Vasey took part, and the measurements were repeated or checked by L. F. Richardson. The following table shows the means.

Table I: Design of coils.

	Outer	Inner
Axial length between mid planes of end turns, cm.	$66{\cdot}8_6$	$40{\cdot}2_2$
Radius of cylinder through axis of wire, cm. ...	$5{\cdot}45_9$	$3{\cdot}845 \pm 0{\cdot}001$
Turns per centimetre of axis	$12{\cdot}8_3$*	$8{\cdot}27_8$
Diameter of copper wire, cm.	$0{\cdot}06_3$ over enamel	$0{\cdot}056$
Insulation	Enamel and cotton, shellaced	Double silk, shellac and celluloid

* Mean of two decimetre portions near ends of inner coil. Middle only $12{\cdot}5_3$, general mean about $12{\cdot}7$.

It is found from these measurements that

$$\theta_A = 22^\circ\ 18'{\cdot}5,\ \theta_B = 180^\circ - 15^\circ\ 12'{\cdot}3_3,\ \theta_C = 180^\circ - 15^\circ\ 8'{\cdot}9,\ \theta_D = 180^\circ - 5^\circ\ 49'{\cdot}2,$$
$$\cos\theta_A - \cos\theta_B - \cos\theta_C + \cos\theta_D = 1{\cdot}860,$$

Messrs S. O. Myers, D. J. Proctor, W. Toyne and L. F. Richardson agreeing in 1·860.

Hence formula (3) gives the intensity at the centre of the end of the inner coil when 1 ampere flows in the outer coil as

$$H_1 = 14{\cdot}99 \text{ gauss} \qquad \text{......(9).}$$

Hence, from (8), as $g = 981{\cdot}2$ cm.sec.$^{-2}$,

$$2m = 235{\cdot}_0 j^2 \text{ when } j \text{ is in E.M.U.}$$

That is to say for one ampere the mass to balance reversal would, from length measurements, be $2{\cdot}35_0$ grams, according to this first approximation ......(10).

The flexible leading wires to the suspended coil are No. 43 S.W.G. tinned copper, one strand sufficing each way.

The resistance of the instrument is about 34 ohms.

The balance was obtained from Messrs F. E. Becker, London, who list it as carrying 300 grams, with a sensitivity of 3 to 4 milligrams, and a price of £3. 7*s*. 6*d*. The stirrups at the ends of the beam have arrestments; without these we have had trouble. The beam lengths agree to 1 in 10,000. The balance is in a glass case; the coils in a cupboard underneath.

§ 4. BEHAVIOUR

It was found at once that the weight to balance reversal of one ampere was about 2·3 grams so that predictions were roughly correct; but the following three difficulties had to be met:

(i) The suspended system decreased in weight during a run. Not all of this decrease was regained on cooling. So presumably the permanent part is due to evaporation, the temporary part to rising air-currents and to softening of the leading-in wires. The effect was as much as 0·1 or 0·2 gram. The observations were arranged so as to eliminate it by allowing an hour or more for preliminary warming, by repeated reversal of the current in one coil only, and by weighings made at observed times.

(ii) The oscillations of the balance were at first irregular, when the coils were connected to accumulators. Thus, choosing as example an unusually violent disturbance, successive turning points were

13·3		17·0		15·5	
	8·4		8·6		9·9

This effect is probably due to an accumulation of hot air in the coils coming out in a sudden upset, instead of in a steady flow. This happened when the outer tube was closed below and open above. After fitting a lid to the outer tube, oscillations were much steadier, successive turning points being for example

13·5		13·0		12·6		12·1		11·8		11·5		11·3	
	9·5		9·6		9·8		9·9		9·9		10·0		10·2

(iii) Fluctuations of $\pm$ 0·001 ampere in the current persisted. Opening the door of the coil-case increased the current suddenly by more than 0·003 ampere; so it is probable that the fluctuations are due to eddying air currents cooling the coils. Accordingly the current was adjusted frequently, and a record kept of its wandering.

Specimen observation

1928 Sept. 17. Current flowing for 4 hours before weighings began. The current was adjusted until the drop of potential across 1 ohm just balanced a standard cell, momentarily put in circuit by a tapping key. Standard Resistance, Tinsley No. 15,521 to carry 3 amperes, oil cooled, certified by the makers to be 1·0001 ohms at 26° C. Standard Cell, by Cambridge Inst. Co., certified by N.P.L. to give 1·0183 international volts at 20° C. One cm. motion of the galvanometer-spot corresponded to a change of 10^{-4} ampere.

Observer L.F.R.

Table II: Variation of readings with time.

Time mins.	Temp. of ohm	Galv. deflection cm.	Switch	Weight in pan grm.	Turning points			Centre of oscil.	Weight to centre
2	—	0, + 5	→	13·33	6·6	11·6	6·6	9·1	13·322
5	25·7	0, − 3	→	13·32	15·0 6·2	5·8 14·2	14·4 6·0	10·2	
9?	—	0, + 8	←	10·88	12·7 7·0	7·2 12·5	13·1 7·2	9·9	10·879
12	25·8	0, ± 3	←	10·87	13·1 8·2	8·1 13·0	13·3 8·7	10·7	
17	—	0, − 4	→	13·31	13·9 4·9	4·8 13·6	13·7 5·2	9·3	13·303
20	—	± 5	→	13·30	8·3 12·2	12·4 8·3	8·4 12·2	10·3	
24	25·8	± 3, + 3	←	10·86	11·4 8·8	8·6 10·8	10·9 8·4	9·8	10·857
26	—	± 2	←	10·85	7·8 12·8	12·8 8·1	8·1 12·8	10·4	
31	—	± 4, − 10	→	13·29	4·2 13·4	13·7 4·9	4·9 12·8	9·1	13·277
34	25·9	± 3	→	13·28	7·7 11·3	11·6 8·8	7·6 11·1	9·8	
39	—	± 3, − 2	←	10·84	9·8 9·2	9·2 9·9	10·1 10·3	9·7	10·835
41	—	± 2, − 4	←	10·83	9·3 11·6	11·6 9·2	9·0 11·1	10·3	
46	25·9	± 3, + 3	→	13·27	5·7 12·4	12·5 5·9	5·5 12·2	9·1	13·266
52	—	± 5, + 5	→	13·26	11·7 10·4	10·7 11·6	11·7 10·4	11·1	

Temperature of Weston cell 20°·8 C.
Temperature of standard cell 20°·6 C.

Now if we assume a steady drift of 1·4 milligrams per minute and correct all the weights to 24^{min} we obtain

						Mean
switch	→	13·292	13·294	13·288	13·300	13·293
	←	10·859	10·857	10·856	—	10·857
					Difference	2·436 ± 0·005 grm.

The mean electric current was 1·0183 1·0001 international ampere. Therefore to balance reversal of one ampere required

$$2{\cdot}35_0 \text{ grams.}$$

Other observations made without a lid on the outer coil, were as follows:

Table III: Variation with time without lid on outer coil.

Date	Drift, milligrams per minute	Grams to balance reversal of one internat. ampere	Observers
Feb. 4	—	$2{\cdot}34_8$	L. F. Richardson
Feb. 7	1·2	$2{\cdot}35_0$	„
Feb.	—	$2{\cdot}33_8$	C. C. Prior and W. L. Vasey
Sept. 15	2·8	$2{\cdot}36_6$	L. F. Richardson
	Mean	2·350	

§ 5. SECOND APPROXIMATION

Take rectangular coordinates x, y, z, the z-axis coinciding with the axis of the coils, and the origin being in the plane of the end of the inner coil. Let the magnetic intensity have components H_X, H_Y, H_Z.

The first approximation consisted in the assumption that H_Z was uniform over the end of the inner coil. For the second approximation expand H_Z in powers of x and y as far as the second degree. By symmetry the linear terms are absent, together with xy, so that

$H_Z = A + B(x^2 + y^2)$ in which A and B are independent of position ......(11).

Let $x^2 + y^2 = r^2$. The flux of induction into one end of the inner coil, which has been denoted in (6) by $N/2$ is therefore

$$\frac{N}{2} = \int_0^{r_2} H_Z \, 2\pi r \, dr = A\pi r_2^2 + B 2\pi \int_0^{r_2} r^3 dr = \pi r_2^2 A \left(1 + \frac{B}{2A} r_2^2\right) \quad \text{......(12).}$$

The value of A has already been found; see (3) and (6).

Now we can find B from a general property of the magnetic field. For, except in the electric current, a magnetic potential ϕ exists such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{......(13),}$$

and

$$H_Z = -\partial\phi/\partial z \quad \text{......(14).}$$

Differentiating (13) with respect to z

$$\frac{\partial^2 H_Z}{\partial x^2} + \frac{\partial^2 H_Z}{\partial y^2} + \frac{\partial^2 H_Z}{\partial z^2} = 0 \quad \text{......(15).}$$

But by (11) we have

$$\frac{\partial^2 H_Z}{\partial x^2} = \frac{\partial^2 H_Z}{\partial y^2} = 2B \quad \text{......(16).}$$

So, from (15) and (16),

$$B = -\frac{1}{4} \frac{\partial^2 H_Z}{\partial z^2} \quad \text{......(17).}$$

And $\partial^2 H_Z/\partial z^2$ can be found on the axis from (3) in which

$$\cos\theta = z(r_1^2 + z^2)^{-\frac{1}{2}} \text{ for each suffix } A, B, C, D \quad \text{......(18),}$$

z being the distance of the appropriate end of the outer coil. Differentiating (18) twice with respect to z, and inserting in (3) it is found that

$$\frac{\partial^2 H_Z}{\partial z^2} = 2\pi n_1 j \left(\frac{-3}{r_1^2}\right) \{\cos\theta_A (\sin\theta_A)^4 - \cos\theta_B (\sin\theta_B)^4 - \cos\theta_C (\sin\theta_C)^4 + \cos\theta_D (\sin\theta_D)^4\} \quad \text{......(19).}$$

Thus the correcting factor $1 + Br_2^2/(2A)$ in (12) comes to

$$1 + \frac{3}{8}\left(\frac{r_2}{r_1}\right)^2 \frac{\cos\theta_A (\sin\theta_A)^4 - \cos\theta_B (\sin\theta_B)^4 - \cos\theta_C (\sin\theta_C)^4 + \cos\theta_D (\sin\theta_D)^4}{\cos\theta_A - \cos\theta_B - \cos\theta_C + \cos\theta_D} \quad \text{......(20).}$$

And on inserting the dimensions given in § 2 this comes to 1·0028; the weight for a given current being larger in this ratio than the first approximation indicated.

DISCUSSION

Dr E. H. Rayner: I should like to draw attention to the utility of the electromagnetic arrangement of the coils used in the Kelvin Balance for the purpose of physical measurements generally. In the absolute attracted disc voltmeter of the Kelvin type, developed by Palm and made by Hartmann and Braun, in which the active disc is about 2 or 3 cm. in diameter and the working distance about 0·5 cm., the attractive force is balanced by the force produced by current in coils of the type used in the Kelvin Balance; for this purpose a small continuous current is derived from a battery of a few volts and adjusted by means of variable resistance. The main object of this voltmeter is to enable voltages of a few hundred kilovolts to be measured, the whole system being immersed in compressed nitrogen. The observations are made through a telescope and the adjustment is effected by means of a long insulating handle. An accuracy of the order of 1 per cent. is claimed. The forces due to convection currents have been found to be one of the limitations of the accuracy of this type of instrument. The double arrangement, with coils suspended from each arm of the balance, is necessary as thereby the forces are largely balanced.

The President: I should like to ask what percentage accuracy can be easily ensured in weighing a current of about an ampere, and whether the apparatus shown can be recommended for the calibration of ammeters and the like in circumstances where substandards are not available.

Dr L. F. Richardson: (In reply to the President). The accuracy at one ampere is about 0·5 per cent. We have used this balance for standardizing an ammeter. Our main intention however has been to provide an instructional absolute instrument to serve the same purpose for Final students as does the tangent galvanometer for Intermediate students.

(In reply to Dr Rayner's remarks about convection.) We do always eliminate the greater part of the force due to convection by reversal of the current in the inner coil relative to its sense in the outer coil. It is only the irregular turbulent variations in the convection-currents that limit the accuracy. The double arrangement, with coils suspended to each arm of the balance, would double the electromagnetic force, but would be likely also to increase the "standard deviation" of the turbulent force in the ratio $\sqrt{2}$; so that the fractional uncertainty would be reduced only in the ratio $\frac{1}{2}\sqrt{2}$, which is not important in an instructional instrument. But doubling the coils would save time otherwise spent on reversals.

SOME NOTES ON WIRELESS METHODS OF INVESTIGATING THE ELECTRICAL STRUCTURE OF THE UPPER ATMOSPHERE. I*

By E. V. APPLETON, F.R.S.

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ABSTRACT. Various direct wireless methods of measuring the "effective" height of the atmospheric ionized layer are discussed and compared. For a layer of horizontal stratification, and under conditions for which the influence of the earth's magnetic field may be neglected, the general equivalence of the quantities measured by the various methods is demonstrated. The effective height in such cases is shown to be greater than the maximum height reached by the atmospheric ray. Proposals for using these methods to obtain information concerning the gradient of ionization in the layer are put forward.

§1. INTRODUCTION.

Within the last few years various direct wireless methods have been proposed for determining the "effective" height of the ionized layer in the upper atmosphere. There is a somewhat wide divergence in the experimental values obtained by these methods so that it is pertinent to inquire what significance should be attached to the term "effective" in the various cases and also to ascertain whether any agreement is to be expected if the experiments are conducted under similar conditions of wave-length, location, etc. It is easily seen that, if we are dealing with cases of true reflection at a very sharply bounded and highly conducting layer, the track of the atmospheric waves being entirely in unionized air, all the methods should give the same result. If, however, the waves are returned to the ground by the process of "ionic refraction"† in a dispersive medium a more detailed examination is necessary before any corresponding statement can be made. The various methods are therefore discussed in greater detail below.

§2 (*a*). WAVE-LENGTH CHANGE METHOD.

This method was proposed by Appleton and Barnett and used on transmissions from the Bournemouth B.B.C. station received at Oxford‡. More recently it has been developed at the Peterborough Radio Research Station of the Department of Scientific and Industrial Research.

To illustrate the principles of this method let us consider a sending station at A (see Fig. 1) sending to a receiving station at E. In the ideal case of an ionized layer

* Paper read at the Brussels meeting of the Union Radio Scientifique Internationale, Sept. 13, 1928.

† *Vide* Eccles, *Proc. R. S.* A, 87, 79 (1912) and Larmor, *Phil. Mag.* (6), 48, 1025 (1924).

‡ *Nature*, March 7, 1925.

of horizontal stratification two sets of waves are received at E, the ground waves travelling direct along AE and the atmospheric waves travelling via $ABCHE$. Let the optical path difference between these two tracks be D, where D is, in general, a function of the wave-length of the radiation. We then have

$$n = \frac{D}{\lambda} \qquad \text{......(1)},$$

where n is the number of wave-lengths the atmospheric waves are behind the ground waves at the point E and λ is the wave-length *in vacuo*. We have, further,

$$D = \int_{ACE} \mu \, ds - AE \qquad \text{......(2)},$$

where μ is the refractive index at any point and ds is an element of length of the atmospheric ray path*.

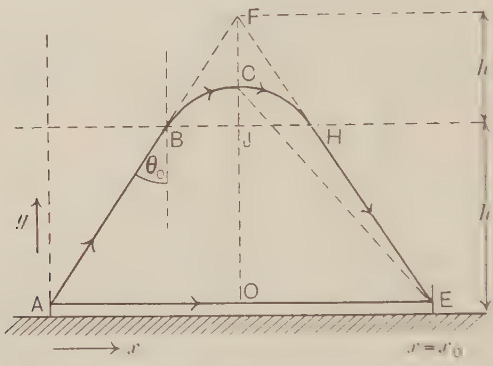

Fig. 1.

The experimental procedure in the wave-length change method consists in measuring the variation in n (as shown by the signal current maxima and minima experienced at E) when λ is continuously varied through a small range. From (1) the variation in n is given by

$$\delta n = -\frac{D}{\lambda^2}\delta\lambda + \frac{1}{\lambda}\delta D \qquad \text{......(3)}.$$

It has been usual to neglect the variation of D in (3) and consider $\lambda^2 \frac{\delta n}{\delta \lambda}$ as the "equivalent path difference" between atmospheric and ground ray paths from which, by simple triangulation, the "equivalent height" of the deviating layer was obtained. The justification of the use of the term "equivalent height" in this connection will be seen in what follows.

It is shown in Appendix 1 that for a variation in optical path due to a finite change of wave-length such as we are considering, we have

$$\delta D = \delta \int_{ACE} \mu \, ds \doteq \int_{ACE} \frac{\partial \mu}{\partial \lambda} ds \, \delta\lambda \qquad \text{......(4)},$$

* The possibility of the dispersion of the ground waves is discussed below.

so that, combining with (2) and (3) we have

$$-\lambda^2 \frac{\delta n}{\delta\lambda} = \int_{ACE} \mu \, ds - \lambda \int_{ACE} \frac{\partial \mu}{\partial \lambda} ds - AE \qquad \text{......(5).}$$

But, from the usual relation between group velocity U, refractive index and wave-length we have, as shown in Appendix II,

$$c \int \frac{ds}{U} = \int \mu \, ds - \lambda \int \frac{\partial \mu}{\partial \lambda} ds \qquad \text{......(6),}$$

where c is the velocity of radiation in free space, so that from (5) and (6) we have

$$\lambda^2 \frac{\delta n}{\delta\lambda} = c \int_{ACE} \frac{ds}{U} - AE \qquad \text{......(7),}$$

where the negative sign of the experimentally determined quantity $\lambda^2 \frac{\delta n}{\delta\lambda}$ has been omitted since δn is negative for an increase of wave-length. We thus see that the "equivalent path difference" is equal to the product of the velocity in free space multiplied by the temporal retardation of a signal traversing the atmospheric path with its appropriate group velocity compared with a signal traversing the ground path. This is quite a different quantity from the actual path difference obtained from (3) by neglecting δD for which approximation we should write, using (2),

$$\lambda^2 \frac{\delta n}{\delta\lambda} = \int_{ACE} \mu \, ds - AE \qquad \text{......(7 a).}$$

In the case in which the influence of the earth's magnetic field may be neglected (e.g. with short waves) it is easy to show that $U = c\mu$ so that, for such conditions the optical path of the atmospheric ray would be $\int \mu \, ds$ while the equivalent path would be $\int \frac{ds}{\mu}$.

It should be mentioned here that, strictly speaking, we should also write

$$\text{Ground Path} = \int_{AOE} \mu \, ds \qquad \text{......(8),}$$

so that for a variation in wave-length

$$\delta\,(\text{Ground Path}) = \int_{AOE} \frac{\partial \mu}{\partial \lambda} ds \, \delta\lambda \qquad \text{......(9),}$$

leading ultimately to the general result

$$\lambda^2 \frac{\delta n}{\delta\lambda} = c \int_{ACE} \frac{ds}{U} - c \int_{AOE} \frac{ds}{U'} \qquad \text{......(10),}$$

where U' is the group velocity of the waves along the ground. It seems quite possible that U' may differ appreciably from c when very short waves are used.

§2 (*b*). ANGLE OF INCIDENCE METHOD.

This method was proposed by Appleton and Barnett* and used at Cambridge to determine the angle of incidence of down-coming waves emitted by the London B.B.C. transmitter. It has also been used by Smith-Rose and Barfield† in a comprehensive study of down-coming waves received from various stations. When the angle of incidence is known the "effective height" of the layer deviating the waves is found by simple triangulation. This quantity would correspond to FO in Fig. 1.

In the general case of magneto-ionic refraction in which the influence of the earth's magnetism is taken into account it is difficult to interpret these experiments much further. If, however, we neglect this influence and also assume that the layer is horizontally stratified a correlation with the wave-length change method may be effected. Since, in such a case, the experimentally determined angle of incidence θ_0 at the ground is also equal to the angle of incidence at the layer we may write

$$\mu \sin\theta = \sin\theta_0 \qquad \text{......(11)},$$

where θ is the angle the atmospheric ray makes with the vertical at any point at which the index of refraction of the medium is μ. Also, as before, we have

$$U = c\mu \qquad \text{......(12)}.$$

Thus, since $dx = ds \sin\theta$ for the atmospheric ray, we have, using (11) and (12)

$$c\int\frac{ds}{U} = \int\frac{ds}{\mu} = \int\frac{dx}{\sin\theta_0} = (AF + FE) \qquad \text{......(13)}.$$

The quantity $c\int\frac{ds}{U}$, as Breit and Tuve‡ first pointed out, is therefore equal to the fictitious path AFE (see Fig. 1). We thus see that, in the case of short waves, the equivalent path of the atmospheric waves measured by the wave-length change method is equal to the effective path obtained by simple triangulation using the angle of incidence of the waves as measured at the ground. Thus both methods (*a*) and (*b*) give the height FO which is greater than the true height CO (see Fig. 1).

§2 (*c*). GROUP-RETARDATION METHOD.

This method was first proposed and used by Breit and Tuve§ who used a transmitter sending out short pulses of radio-frequency energy and measured the time interval Δt between the arrival of the signal pulses received via the ground and via the atmosphere.

For such a case, using again Fig. 1 we may write, simply,

$$\Delta t = \int_{ACE}\frac{ds}{U} - \int_{AOE}\frac{ds}{U'} \qquad \text{......(14)},$$

from which the general equivalence of what is measured in the wave-length change method and in the group retardation method is at once apparent.

* *Electrician*, April 3, 1925, p. 398 and *Proc. R. S.* A, **109**, 621 (1925).
† *Proc. R. S.* A, **110**, 580 (1926) and **116**, 682 (1927).
‡ *Physical Review*, **28**, No. 3, 571 (1926).
§ *Loc. cit.*

It has, however, been pointed out to me by Mr J. A. Ratcliffe that (14) applied to the group-retardation method requires qualification, since the component frequencies of the group do not follow the same path. The problem is therefore a little more complicated than that contemplated in the ordinary discussions of group velocity in which the frequency components (usually two) of the group travel along the same (usually straight) path. The following justification of the applicability of (14) is therefore offered. The case of a particularly simple type of group transmission which has not yet been tried in practice is first examined. The discussion of the actual type of group transmission used by Breit and Tuve then follows.

Let us consider the simplest case of the transmission of a group of waves and imagine two sets of waves, the frequencies of which differ only slightly, emitted by a sending station. If the angular frequencies of the waves are $p - \delta p$ and $p + \delta p$ beats of frequency $\frac{\delta p}{\pi}$ will be produced when the waves travel in unionized air along the same track. For both sets of waves to arrive at the same receiving station (e.g. E in Fig. 1) they must set out from the sending station along slightly different directions because of the dispersion in the ionized layer. To allow for the difference in frequency and for the difference in the optical paths of the two components we say that the received signal amplitude is proportional to

$$\cos(p - \delta p)\left(t - \frac{P - \delta P}{c}\right) + \cos(p + \delta p)\left(t - \frac{P + \delta P}{c}\right) \quad \text{......(15)},$$

or to

$$2\cos p\left(t - \frac{P}{c}\right)\cos\left(\frac{\delta p P}{c} + \frac{p\,\delta P}{c} - t\,\delta p\right) \quad \text{......(16)},$$

where P is the optical path of the atmospheric ray for angular frequency p. The expression for the received signal amplitude represents a high frequency oscillation of slowly varying amplitude. The particular maximum amplitude emitted by the sending station when t and P are zero is reproduced again at the receiver after a time t given by

$$\frac{\delta p P}{c} + \frac{p\,\delta P}{c} - t\,\delta p = 0,$$

or by

$$ct = \int \mu\, ds + p\frac{\delta}{\delta p}\int \mu\, ds \quad \text{......(17)}$$

$$= \int \mu\, ds - \lambda\frac{\delta}{\delta\lambda}\int \mu\, ds \quad \text{......(18)}.$$

Now from Appendix I we have

$$\frac{\delta}{\delta\lambda}\int \mu\, ds = \int \frac{\partial\mu}{\partial\lambda}\, ds.$$

Thus (18) becomes

$$ct = \int \mu\, ds - \lambda\int \frac{\partial\mu}{\partial\lambda}\, ds \quad \text{......(19)}$$

$$= c\int \frac{ds}{U} \quad \text{......(20)}.$$

We thus see that a particular beat maximum emitted by the sending station is produced again at the receiving station after a time $\int \frac{ds}{U}$ where U is the group

velocity for the mean frequency. The use of (14) is therefore justified even though the two component frequencies have followed slightly different tracks.

To measure the group retardation in such an experiment as has been suggested above, the down-coming waves could be received on a suppressed ground ray system and the high frequency oscillations produced rectified, thus transforming the beats into combination tones. The phase of the combination tone received via the atmosphere could be compared with that of the combination tone received via the ground waves and, from a knowledge of the beat frequency, the retardation found.

An imaginary experiment shows even more closely the essential equivalence of the wave-length change method and the particular type of group retardation method described immediately above. Let us suppose that one of the emitted wave-lengths is maintained constant and the other made gradually different so that the beat frequency increases from zero. The phase difference between the beat amplitudes received via the atmosphere and via the ground will gradually increase. Let us now suppose that when the wave-lengths differ by $\delta\lambda$ the phase difference is ϕ where ϕ is $2\pi n'$. The group retardation Δt is then evidently $n't'$, where t' is the time of one beat*. But t' is equal to $\frac{\lambda^2}{c\delta\lambda}$, so that, using again Fig. 1, we have

$$\Delta t = \int_{ACE} \frac{ds}{U} - \int_{AOE} \frac{ds}{U'} = n' \left(\frac{\lambda^2}{c\delta\lambda}\right) \qquad \text{......(21).}$$

But if the two wave-lengths emitted by the sending station had been the initial and final wave-lengths during a wave-length change experiment we should have from (10)

$$\lambda^2 \frac{\delta n}{\delta\lambda} = c\int_{ACE} \frac{ds}{U} - c\int_{AOE} \frac{ds}{U'} \qquad \text{......(22),}$$

so that

$$n' = \delta n.$$

Thus the number of "fringes" measured in the wave-length change method is equal to the number of beats the atmospheric ray is behind the ground ray when the beats are produced by the simultaneous emission of the initial and final wave-lengths used.

The extension to the case of pulses of radio-frequency waves as actually used by Breit and Tuve follows on similar lines. Let us suppose that the group may be represented by the sum of a large number of sinusoidal vibrations the frequencies of which differ only slightly from $p/2\pi$. The disturbance arriving at the receiving station may then be represented by

$$\Sigma a \cos (p + \delta p)\left(t - \frac{P + \delta P}{c}\right),$$

which is equal to

$$\cos p\left(t - \frac{P}{c}\right) \Sigma a \cos\left(t\delta p - \delta p \frac{P}{c} - p\frac{\delta P}{c}\right)$$

$$- \sin p\left(t - \frac{P}{c}\right) \Sigma a \sin\left(t\delta p - \delta p \frac{P}{c} - p\frac{\delta P}{c}\right).$$

* In other words, n' is the time interval between the beats as received via the atmosphere and via the ground, expressed in terms of a beat period as unit of time.

We thus get the reproduction of the original wave-form (i.e. the group is received) at the receiver when

$$t\delta p - \delta p \frac{P}{c} - p \frac{\delta P}{c} = 0,$$

or when

$$t = \frac{1}{c}\left(\int \mu\, ds + p \frac{\delta}{\delta p} \int \mu\, ds\right) \qquad \text{......(23).}$$

But (23) is identical with (17) which was shown to lead to

$$t = \int \frac{ds}{U} \qquad \text{......(24).}$$

In this case t is the time of transmission of the group through the atmosphere and U the group velocity of the mean frequency. In the actual experiment of Breit and Tuve the use of (14) may therefore be justified even when the different components of the group follow different tracks in the upper atmosphere.

It will be seen that, for the measurement of the group-retardation, it is not necessary to send out "chopped"* signals like those used by Breit and Tuve. A sinusoidally-modulated wave would suffice for receiving stations at which the ground waves are strong. In the case of short waves and fairly long transmission distances the ground waves would be strongly absorbed and thus of negligible strength at the receiver. In such cases the modulation signal or the "chopped" signals of Breit and Tuve could be transmitted simultaneously along the ground by a much longer wireless wave-length or possibly by telephone.

Summarising the results of this section we may say that:

(1) The wave-length change method and the group-retardation method measure the same quantity $\left(\text{i.e. } c\int \frac{ds}{U} \text{ for the atmospheric waves}\right)$.

(2) In cases for which the effect of the earth's magnetic field is negligible and the layer is horizontally stratified, the above-mentioned methods, together with the angle of incidence method, give us the equivalent height FO in Fig. 1. The equivalent height is greater than the actual maximum height reached by the atmospheric waves.

§3. SOME SPECIAL IONIZATION GRADIENTS

It is instructive to consider some of the questions discussed above for some particular types of ionization gradients in the layer.

Let us suppose (see Fig. 2 in which the lettering is the same as in Fig. 1) that a ray enters the layer at B, and that the gradual deviation of it is brought about by the gradual reduction of the refractive index (due to the increasing ionization) experienced as it penetrates the layer. Consider any point P on the path of the ray and let the tangent to the path at that point make an angle θ with the vertical (i.e. with the y axis in the figure). We then have, as before,

$$\mu \sin\theta = \mu_0 \sin\theta_0 = \sin\theta_0 \qquad \text{......(25),}$$

* A simple method of obtaining these short trains of oscillations is to use a transmitter with a high grid leak the circuit being similar to that used for producing the unidirectional cathode-ray oscillograph time base. *Vide Proc. R. S.* A, **111**, 672 (1926).

where μ is the refractive index at the point P and μ_0 (= 1) is the refractive index of the unionized air. θ_0 is the angle of incidence at the layer. The refractive index μ is considered as being independent of x and dependent only on y and the wave-length.

Since
$$\tan\theta = \frac{dx}{dy},$$
and
$$ds = \sqrt{1 + \tan^2\theta}\, dy,$$
the optical path in the medium is given by

$$\int \mu\, ds = 2\int_0^{y_0} \mu \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\int_0^{y_0} \frac{\mu^2}{\sqrt{\mu^2 - \sin^2\theta_0}}\, dy \qquad \text{......(26)}$$

where $\mu = f(y, \lambda)$ and $\sin\theta_0 = f(y_0, \lambda)$.

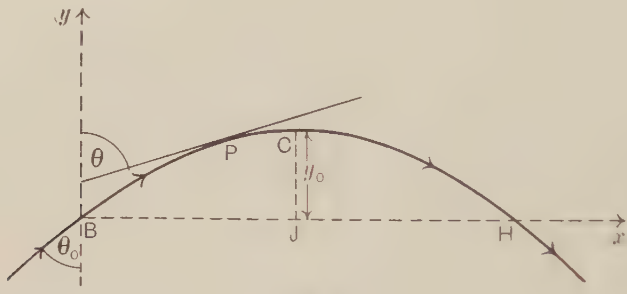

Fig. 2.

Correspondingly we have for the "equivalent" path

$$\int \frac{ds}{\mu} = 2\int_0^{y_0} \frac{dy}{\sqrt{\mu^2 - \sin^2\theta_0}} \qquad \text{......(27).}$$

The above expressions have been derived on the assumption that μ is a single-valued function of position for constant wave-length and so the argument only applies to the case in which the influence of the earth's magnetic field is negligible. We therefore consider the case as applying to the propagation of short waves. For this reason we may take simple types of variation of the refractive index such as

$$\mu^2 = 1 - \alpha y \qquad \text{......Case } (\alpha),$$

and

$$\mu^2 = 1 - \beta y^2 \qquad \text{......Case } (\beta).$$

Case (α) corresponds to a gradient in which the ionization varies directly with the height above the lower boundary of the layer, while Case (β) corresponds to ionization varying as the square of the height. By substituting for μ^2 in (25) and (26) and integrating it is possible to arrive at the following results for the two cases.

Case (α). $\mu^2 = 1 - \alpha y$:

$$\left.\begin{aligned}
&\text{Optical path in the layer} &&= \int \mu\, ds = \frac{4}{\alpha}\cos\theta_0 - \frac{8}{3\alpha}\cos^3\theta_0\\
&\text{Equivalent path in the layer} &&= \int \frac{ds}{\mu} = \frac{4}{\alpha}\cos\theta_0\\
&\text{Horizontal distance in the layer } (BH \text{ in figure}) &&= \frac{4}{\alpha}\sin\theta_0 \,.\, \cos\theta_0\\
&\text{Maximum height in the layer } (CJ \text{ in figure}) &&= \frac{\cos^2\theta_0}{\alpha}
\end{aligned}\right\} \quad \ldots\ldots(28).$$

Case (β). $\mu^2 = 1 - \beta y^2$:

$$\left.\begin{aligned}
&\text{Optical path in the layer} &&= \int \mu\, ds = \frac{\pi}{2\sqrt{\beta}}(1 + \sin^2\theta_0)\\
&\text{Equivalent path in the layer} &&= \int \frac{ds}{\mu} = \frac{\pi}{\sqrt{\beta}}\\
&\text{Horizontal distance in the layer} &&= \frac{\pi \sin\theta_0}{\sqrt{\beta}}\\
&\text{Maximum height in the layer} &&= \frac{\cos\theta_0}{\sqrt{\beta}}
\end{aligned}\right\} \quad \ldots\ldots(29).$$

Now in attempting to verify for the above examples that the equivalent path $\int \frac{ds}{\mu}$, which is the quantity measured in the group-retardation method, is the same as the quantity $\int \mu\, ds - \lambda \frac{\delta}{\delta\lambda} \int \mu\, ds$ measured in the wave-length change method, we have to consider the problem as a whole and recognise that both the angle of incidence at the layer and the track in and out of the layer all vary when we vary the wave-length. Let us therefore consider the complete problem of a layer situated at a distance h above the ground (see Fig. 1) and take the case of transmission between two points a distance d (i.e. AE in Fig. 1) apart. For stationary conditions of wave-length we can write in the two cases:

Case (α):

$$\left.\begin{aligned}
&\text{Total optical path of atmospheric ray} &&= \int \mu\, ds = \frac{2h}{\cos\theta_0} + \frac{4}{\alpha}\cos\theta_0 - \frac{8}{3\alpha}\cos^3\theta_0\\
&\text{Total equivalent path of atmospheric ray} &&= \int \frac{ds}{\mu} = \frac{2h}{\cos\theta_0} + \frac{4}{\alpha}\cos\theta_0\\
&\text{Path of ground ray} &&= d = 2h\tan\theta_0 + \frac{2}{\alpha}\sin 2\theta_0
\end{aligned}\right\} \quad \ldots\ldots(30).$$

Case (β):

$$\left.\begin{aligned}
&\text{Total optical path of atmospheric ray} &&= \int \mu\, ds = \frac{2h}{\cos\theta_0} + \frac{\pi}{2\sqrt{\beta}}(1 + \sin^2\theta_0)\\
&\text{Total equivalent path of atmospheric ray} &&= \int \frac{ds}{\mu} = \frac{2h}{\cos\theta_0} + \frac{\pi}{\sqrt{\beta}}\\
&\text{Path of ground ray} &&= d = 2h\tan\theta_0 + \frac{\pi \sin\theta_0}{\sqrt{\beta}}
\end{aligned}\right\} \quad \ldots\ldots(31).$$

Now the recognition of the essential equivalence of the wave-length change method and the group-retardation method was based on the proof that, for such cases as we are considering $\int \mu\, ds - \lambda \frac{\delta}{\delta\lambda} \int \mu\, ds$ was equal to the equivalent path $\int \frac{ds}{\mu}$. In attempting to demonstrate this for these particular cases we note that, in finding the value of the expression $\frac{\delta}{\delta\lambda} \int \mu\, ds$, we require to know both $\frac{\partial \alpha}{\partial \lambda}$ $\left(\text{or } \frac{\partial \beta}{\partial \lambda}\right)$ and $\frac{\partial \theta_0}{\partial \lambda}$, for the expression for $\int \mu\, ds$ involves both these quantities. Now in an ionized medium in which the frequency of the electronic collisions with the gas molecules is small, we may write

$$\mu^2 = 1 - A\lambda^2 \qquad \text{......(32)},$$

where A is a quantity* which is proportional to the ionization. Thus for our two cases we have

Case (α) $\qquad \mu^2 = 1 - \alpha y = 1 - B\lambda^2 y,$

Case (β) $\qquad \mu^2 = 1 - \beta y^2 = 1 - C\lambda^2 y^2,$

where B and C are constants. From these expressions $\frac{\partial \alpha}{\partial \lambda}$ and $\frac{\partial \beta}{\partial \lambda}$ may be calculated. Also, from the expressions for the total path of the ground ray in each case we can find $\frac{\partial \theta_0}{\partial \lambda}$ after making the substitution for $\frac{\partial \alpha}{\partial \lambda}$ or $\frac{\partial \beta}{\partial \lambda}$ as the case may be. In this way we arrive at the value of $\int \mu\, ds - \lambda \frac{\delta}{\delta\lambda} \int \mu\, ds$ for the total atmospheric paths in the two cases. These are found to be

Case (α)
$$\int \mu\, ds - \lambda \frac{\delta}{\delta\lambda} \int \mu\, ds = \frac{2h}{\cos\theta_0} + \frac{4}{\alpha}\cos\theta_0 \qquad \text{......(33)},$$

Case (β)
$$\int \mu\, ds - \lambda \frac{\delta}{\delta\lambda} \int \mu\, ds = \frac{2h}{\cos\theta_0} + \frac{\pi}{\sqrt{\beta}} \qquad \text{......(34)}.$$

But from (30) and (31) we see that these expressions are exactly those for the equivalent paths. We therefore have verified the statement that in the wave-length change method we measure the equivalent path $\int \frac{ds}{\mu}$ and not the optical path $\int \mu\, ds$.

From the point of view of deducing information concerning α and β from experimental determinations of $c\,\Delta t$ or $\lambda^2 \frac{\delta n}{\delta\lambda}$ we note that the relevant equations are as follows:

Case (α)
$$c\,\Delta t = \lambda^2 \frac{\delta n}{\delta\lambda} = \frac{2h}{\cos\theta_0} + \frac{4}{\alpha}\cos\theta_0 - d \qquad \text{......(35)}.$$

Case (β).
$$c\,\Delta t = \lambda^2 \frac{\delta n}{\delta\lambda} = \frac{2h}{\cos\theta_0} + \frac{\pi}{\sqrt{\beta}} - d \qquad \text{......(36)}.$$

* A is actually equal to $\frac{Ne^2}{\pi mc^2}$ where N is the number of electrons, of charge e and mass m, per c.c.

§4. METHODS OF INVESTIGATING THE IONIZATION GRADIENT

We have seen that for short waves any one of the three methods discussed above will enable us to find the value of $\int \frac{ds}{\mu}$ for the atmospheric ray. This quantity is equal to the length AFE in Fig. 3, from which, knowing d, we may find $\sin \theta_0$ which is equal to the refractive index at the point C. Now since, for short waves, we have

$$\mu^2 = 1 - \frac{4\pi N e^2}{m p^2} \qquad \text{......(37)},$$

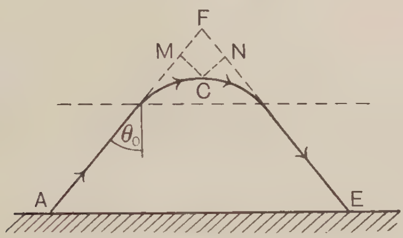

Fig. 3.

a knowledge of the refractive index enables us to calculate the electron concentration N at the point C. If we could measure the actual height of point C in some other way we should be able to specify the ionization at a particular height.

Let CM and CN (in Fig. 3) be perpendiculars from C on AF and FE respectively. Now Pedersen has shown* that the optical path $\int \mu\, ds$ of the atmospheric ray, when $\theta_0 > 30°$, is approximately (actually slightly smaller than) $AM + NE$. If, therefore, it were possible to determine both $\int \frac{ds}{\mu}$ and $\int \mu\, ds$ for the atmospheric ray we should be able to find both the ionization at and height of point C. The equivalent path would fix for us the position of F and enable us to calculate the ionization at C, while the optical path would fix the positions of M and N from which the position of C could be found. To a method of estimating the optical path of the atmospheric ray we now turn.

It will be found convenient to re-write (1) in terms of the frequency f instead of the wave-length λ. Thus

$$n = \frac{fD}{c}$$

and

$$\frac{dn}{df} = \frac{D}{c} + \frac{f}{c}\frac{dD}{df} \qquad \text{......(38)}.$$

Now it is possible to carry out experiments in which $\frac{dn}{df}$ is measured for a large number of frequencies at about the same time and under the same conditions of transmission distance, etc. This quantity would, in general, be a function of the frequency and may be written $\phi(f)$. Thus (38) becomes

$$\frac{dD}{df} + \frac{D}{f} = \frac{c}{f}\phi(f) \qquad \text{......(39)}.$$

* *The Propagation of Radio Waves*, Copenhagen, 1927, p. 176.

the solution of which is given by

$$Df = c \int \phi(f)\, df + \text{const.} \qquad \text{......(40).}$$

We thus see that, even if $\phi(f)$ is known for a considerable range of frequencies, we cannot find D exactly for any particular frequency since the constant is unknown. If, however, after having obtained values of $\phi(f)$ for as low frequencies as we can in practice, we draw the curve showing the relation between $\phi(f)$ and f and extrapolate it to the limit of zero frequency we can get over this difficulty and write

$$D_0 f_0 = c \int_0^{f_0} \phi(f)\, df \qquad \text{......(41),}$$

where D_0 is the optical path difference for a particular frequency f_0. The optical path of the atmospheric ray for the very high frequencies could then be found and the ionization at certain particular heights determined by the method outlined above.

It should be noted that the measurements of $\frac{dn}{df}$ would have to be made at a considerable distance from the transmitter to fulfil the conditions under which the relation of Pedersen holds, and it is most probable that, as the frequency is increased to the values for which the influence of the earth's magnetic field is negligible, the strength of the ground waves at the receiving station would be too small to detect. In such cases the suggestion made above of using at the same time a long wave-length to send out the modulation (e.g. pulses of Breit and Tuve) might be useful.

The method described above deals with measurements made at one receiving station with different mean frequencies. It is, however, possible that information of similar character might be obtained by making simultaneous observations on the same mean frequency at different distances as the following considerations show.

We have seen that any one of the three direct methods considered will enable us to find the equivalent height OF (i.e. $h + h'$ in Fig. 1). Considering again the cases of special types of ionization gradients we find from (35)

$$\lambda^2 \frac{\delta n}{\delta \lambda} + d = \frac{2(h+h')}{\cos\theta_0} = \frac{2h}{\cos\theta_0} + \frac{4\cos\theta_0}{\alpha},$$

or

$$h + h' = h + \frac{2}{\alpha}\cos^2\theta_0 \qquad \text{......(42),}$$

and similarly from (36)

$$h + h' = h + \frac{\pi\cos\theta_0}{2\sqrt{\beta}} \qquad \text{......(43).}$$

We thus see that, for a case of ionization varying directly with the height, the equivalent height should vary linearly with $\cos^2\theta_0$ for observations made at various distances, whereas for ionization varying as the square of the height the equivalent height should vary linearly with $\cos\theta_0$. Corresponding equations for other types of gradient could be found and by comparing the equivalent heights at different distances we should be able to ascertain which type of gradient gives the closest approximation. When this decision has been made the observations from any two stations would enable us to find both h and α (or β) from which the ionization at

definite heights could be stated. Additional evidence could be obtained by making the experiments with different wave-lengths since α (or β) depends on the wave-length. For example, once the type of gradient had been determined, it would also be possible to find h and α (or β) from observations made at any one station on two different wave-lengths. The results could therefore be checked.

In a later communication I hope to deal with the more practical aspects of some of the theoretical points discussed in the present paper. These problems have arisen in connection with investigations carried out for the Radio Research Board of the Department of Scientific and Industrial Research.

APPENDIX I

To find the variation in the optical path of the atmospheric ray

Consider Fig. 1 in which the track of the atmospheric ray $ABCHE$ begins at the origin A and finishes at E $(x_0, 0)$. We have

$$\int \mu \, ds = \int_0^{x_0} \mu \frac{ds}{dx} dx \qquad \text{......(1')}.$$

Now if the refractive index is (as in the most general case of the magneto-ionic theory) a function of position, ray-direction and wave-length we write

$$\mu = f(x, y, z, \lambda, y', z') \qquad \text{......(2')}.$$

Thus

$$\begin{aligned}
\delta \int \mu \, ds &= \delta \int_0^{x_0} \mu \frac{ds}{dx} dx \\
&= \int_0^{x_0} \delta\mu \frac{ds}{dx} dx + \mu\delta\left(\frac{ds}{dx}\right) dx \\
&= \int_0^{x_0} \left\{\frac{\partial\mu}{\partial y}\delta y + \frac{\partial\mu}{\partial z}\delta z - \frac{d}{dx}\left(\frac{\partial\mu}{\partial y'}\right)\delta y - \frac{d}{dx}\left(\frac{\partial\mu}{\partial z'}\right)\delta z + \frac{\partial\mu}{\partial\lambda}\delta\lambda\right\}\frac{ds}{dx} dx \\
&\qquad + \int_0^{x_0} \mu\delta\left(\frac{ds}{dx}\right) dx \qquad \text{......(3')}.
\end{aligned}$$

But by the principle of least time, as interpreted by Hamilton, the terms on the right-hand side representing path variations vanish, leaving only the term involving the wave-length variation. Thus

$$\begin{aligned}
\delta \int \mu \, ds &= \int_0^{x_0} \frac{\partial\mu}{\partial\lambda}\delta\lambda \frac{ds}{dx} dx \\
&= \int \frac{\partial\mu}{\partial\lambda} ds\,\delta\lambda \qquad \text{......(4')}.
\end{aligned}$$

APPENDIX II

The relation between phase and group velocities

The relation between group velocity U and phase velocity V is given by

$$\frac{1}{U} = \frac{\partial}{\partial f}\left(\frac{f}{V}\right) \qquad \text{......(1'')},$$

where f is the frequency of the waves. But V is equal to $\frac{c}{\mu}$ so that this becomes

$$\begin{aligned}\frac{1}{U} &= \frac{1}{c}\frac{\partial}{\partial f}(f\mu)\\ &= \frac{1}{c}\left(\mu + f\frac{\partial \mu}{\partial f}\right)\\ &= \frac{1}{c}\left(\mu - \lambda\frac{\partial \mu}{\partial \lambda}\right) \qquad \text{......(2'')}.\end{aligned}$$

Thus for any particular path we have

$$c\int\frac{ds}{U} = \int \mu\, ds - \lambda\int\frac{\partial \mu}{\partial \lambda}\, ds \qquad \text{......(3'')}.$$

This equation is to be compared with

$$\lambda^2\frac{\delta n}{\delta\lambda} + AE = \int \mu\, ds - \lambda\frac{\delta}{\delta\lambda}\int \mu\, ds \qquad \text{......(4'')},$$

which may be derived from (2) and (3).

In conclusion I wish to express my particular indebtedness to my collaborator Mr J. A. Ratcliffe, who first raised the question of the essential equivalence of the quantities measured in the wave-length change method and in the method of Breit and Tuve by pointing out to me the close similarity between equations (3'') and (4''), which are the representative relations for the two methods. To Mr S. Goldstein I am also greatly indebted for advice on certain mathematical points.

DISCUSSION

Dr J. Hollingworth: The few words I have to say are concerned more with the practical than with the theoretical side of this paper. In the diagrams a fairly small angle of incidence (θ_0) is generally shown corresponding to short distances of transmission, under which conditions the experiments have, I believe, been made. It remains uncertain, however, how far this method would serve for determining whether the effective height is independent of the angle of incidence for larger values of the latter. On page 53 it is specified that $\theta_0 > 30°$ and personally I should be inclined to put this figure definitely higher. My own experience (which, it is true, is derived from a study of very long waves) suggests strongly that for longer distances of transmission the path is definitely of a "flat-topped" form rather than triangular, though decisive evidence is very difficult to obtain.

Mr T. Smith: I am induced to speak on Prof. Appleton's paper because his treatment of the subject involves essentially the same considerations as those which enter into optical problems. Optical experience suggests that if we are measuring the directions in which waves travel, or the statistical distribution of their energy in space, we shall be concerned with $\int \frac{ds}{\mu}$ rather than with the integral $\int \mu\, ds$ from which we start. Almost any measurement we like to make will relate to the former rather than to the latter. In fact it is not at all an easy matter to contrive experiments which depend on $\int \mu\, ds$. For instance a signal of any kind must give $\int \frac{ds}{\mu}$, for a signal is given by imposing a peculiarity of some kind on a train of waves, that is to say it concerns group velocity, not phase velocity. The fact that group velocity is involved is not always obvious; some of the historic methods employed in measuring the velocity of light afford examples in which the distinction is very subtle. The method of measurement at the receiving end has to be carefully considered before we can be sure that it is the phase velocity which is being obtained. For example, consider directional measurement with a monorhythmic train of waves. If this be carried out by cutting off a portion of the wave front our experiments will give the group, not the phase velocity, for in isolating a part of a wave we have impressed a peculiarity on it, and it is this peculiarity we are measuring. It is therefore fortunate that the quantity we are most often interested in is the one it is simpler to obtain. If in optics we desired to evaluate $\int \mu\, ds$ we should fall back upon an interference method, preferably one in which we obtained "white light fringes." Perhaps a corresponding method, in which the time taken by an interfering wave to travel between source and receiver can be controlled and adjusted to give steady agreement of phase with the reflected beam for a range of wave-lengths, would be possible in the wireless problem.

Professor Appleton has shown that $\int \frac{ds}{\mu}$ is related to the length of hypothetical rectilinear paths observed from the ground. We can also relate it to the area enclosed by the wave in its actual path, and the comparison of the two may perhaps give some useful information. Instead of regarding the refracting layers as planes, let us take into account the curvature of the earth: the layers of equal refractive index are then concentric spheres. We may now apply the well-known optical law that the product of the refractive index and the perpendicular to the ray from the centre of the refracting surfaces is constant. Thus $\mu p = P$. Now the area of the triangle having the earth's centre as vertex and the element of path ds as base is $\frac{1}{2} p\, ds$. It follows that

$$\int_A^B \frac{ds}{\mu} = \int_A^B \frac{p\, ds}{\mu p} = \frac{2\Delta}{P},$$

where Δ is the area enclosed between the path AB and the straight lines AG and BG joining A and B to G, the centre of the earth. From the observed height of the point F in Prof. Appleton's diagram the area of the trapezium $AFEG$ can

be found. The difference between this and the area $ABCHEG$ follows, that is to say the area $BFHC$. A superior correction $C'F$ to the height of OF is obtained by assuming that the path in air consists of three straight portions along AF, the horizontal through C' and FE, respectively, the triangle they enclose having its area equal to that found for $BFHC$. A closer approximation can be obtained by assuming that BCH is a parabolic arc. This leads to the conclusion that the area enclosed by AF, FE and the horizontal through C should be equal to three-quarters of the area $BFHC$.

Prof. A. O. RANKINE: There appears to be an analogy between the phenomena discussed in the paper and seismic phenomena which occur both in natural earthquakes and in surface explosions artificially produced, in which the disturbances travel more rapidly in the deeper strata than in those near the surface. In such cases an appreciable fraction of the total disturbance is found to enter the lower medium at the critical angle, travel along the interface with the velocity corresponding to the lower medium, and re-enter the upper medium again at the critical angle. If the distance between the source of disturbance and the point of observation be great enough, the waves following the above path arrive earlier than those proceeding direct through the upper (or lower-velocity) medium. Since the velocity of wireless waves is enhanced in the Heaviside layer, there would appear to be a close, though inverted, analogy between this seismic phenomenon and the travelling of wireless waves in a "flat-topped" path as described by Dr Hollingworth.

THE PRESIDENT: Prof. Appleton has made the subject his own, especially where short distance propagation is concerned. He started his measurements by assuming as a first approximation that the Heaviside layer is well defined and behaves like a mirror, and he obtained valuable information about the layer on that assumption. Now he is inevitably proceeding to a second step in the approximation by examining into the consequences of assuming a transitional space where ionic refraction occurs. The discussion has brought out that the path of the wave may be considerably different from this second approximation, and may even be a very shallow truncated triangle. However, the common point which appears to emerge is that the deflecting layer is indeed rather sharply defined.

AUTHOR'S reply: Dr Hollingworth's deduction, from his long wave measurements, that θ_0 may be taken as being larger than 30° is very helpful, in that it indicates that the first of the two methods of finding the ionization gradient might be carried out with some hope of success. So long as θ_0 is greater than 30°, this method is independent of the path, "flat-topped" or otherwise.

Mr Smith's very interesting result can be used to show how the spherical and plane cases differ. Suppose that, in Fig. 1, AOE is a curved portion of the earth's surface. Then it may be shown that

$$\int \frac{ds}{\mu} = \frac{AOE}{\sin \theta_0} + \frac{2S}{R \sin \theta_0},$$

where S is the area included between the atmospheric path and the ground. The

second term on the right-hand side of this equation is the small correction term (cf. equation 13). If it were possible, by very refined measurements, to find $\int \frac{ds}{\mu}$ and θ_0 accurately, S could be found.

The President has raised the important question of the process by which the atmospheric waves are deviated. Although the paper is based on the ionic refraction theory, it may turn out that the ionization gradient is greater than is commonly supposed and that we shall have to replace simple ionic refraction by some process intermediate between such refraction and true reflection. If, in such a case, the boundary of the layer had a certain degree of "roughness," we could possibly explain "skipped distances" as being regions in which only diffusely-reflected radiation is received, the specularly-reflected radiation being received at greater distances. Such an explanation would give the correct relation between the wave-length and the magnitude of the skipped distance.

SYNDICATE ROOM

THE PHYSICAL INTERPRETATION OF WAVE MECHANICS

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ABSTRACT. The object of this paper is to give an account of the fundamental principles of wave mechanics in a manner which shall make clear the physical significance of all the quantities and processes involved. The principles are illustrated by discussions of the propagation of free electric waves in uniform electromagnetic fields and of bound electric waves in the hydrogen atom. The paper concludes with relativistic wave mechanics (prior to the work of Dirac and Darwin) and a short account of the Compton effect.

CONTENTS

LIST OF SYMBOLS

A	Fermi's coefficient (§ 6·7)
$\mathbf{A}$	electromagnetic vector potential (E.M.U.)
a_1, a_2	Dirac's coefficients (§ 6)
a_n	coefficient in Sonine's polynomial (§ 5·2); radius of Bohr orbit (§ 6·2)
a_{kn}	matrix component of $\mathbf{A}$. grad ψ
$\mathbf{a}$	vector amplitude of $\mathbf{A}$ (§ 7·4)
c	velocity of light
E_{12}	energy radiated in a transition (§ 6·7)
$\mathbf{E}$	electric intensity (E.S.U.)
E	energy of equivalent quantum (§ 7·4)
$-e$	electronic charge (E.S.U.)
F	strength of electrostatic field (§ 4·1)
$\mathbf{f}$	momentum of equivalent quantum (§ 7·4)
$\mathbf{H}$	magnetic intensity (E.M.U.)
h	Planck's constant
L	Lagrangian function
l_1, l_2, l_3	direction cosines
M	mass carried by waves (§ 3·2)
$\mathbf{M}$	magnetic moment of atom
m	mass of electron
$\mathbf{n}$	a unit vector (§ 7·4)
n	a quantum number (§ 5·2)
N	Rydberg constant
p	a positive integer (§ 5·2)
$\mathbf{p}$	momentum of electron (§§ 1, 2)
$\mathbf{p}_{kn}$	polarisation
Q	charge carried by waves
$\mathbf{q}$	velocity of charge carried by waves
$R =$	$\lvert \mathbf{r} - \mathbf{r}' \rvert$ (§ 6·1)
$\mathbf{r}$	position vector
$s =$	$(h/2\pi\iota) \log$ (wave function ψ) (§ 2·2)
$\mathbf{s}$	strength of Hertzian doublet (§ 6)
S	bounding surface
t	time
$T_m^n(n)$	Sonine's polynomial
U	proportional to potential of "quantum force"
V	electrostatic potential (E.S.U.)
$u_1, u_2, \ldots$	wave functions omitting time factor
$\mathbf{v}$	velocity of electron
W	total energy of electron
x, y, z	Cartesian coordinates
z, R, ϕ	cylindrical coordinates
r, θ, ϕ	spherical coordinates
$\boldsymbol{\alpha}$	acceleration of electron (§ 6·5)
α	modulus of ψ (§ 3)
β	proportional to argument of ψ
γ	reciprocal of radius of Bohr's ground orbit
ι	$\sqrt{(-1)}$
κ	parameter measuring concentration of wave packets (§§ 4·2, 4·3) (cms.)
κ	parameter in wave eq. (§ 5) (cms.$^{-1}$)
λ	wave-length (§§ 1, 2)
$\lambda + 1$	azimuthal quantum no. (§ 5·2)
Λ	Lagrangian function for electron (§ 7·1)
μ	equatorial quantum no.
ν	frequency
ρ	volume density of charge (E.S.U.)
σ	current density (E.M.U.)
$d\tau$	element of volume
ψ	wave function
ω	$eH/2mc$, Larmor rotation (§§ 4·3, 5·5)
$\omega =$	$2\pi\iota\nu$ (§ 6)
∇	vector operator

Vectors, $\mathbf{A}$, are printed in Clarendon type. Their components are A_1, A_2, A_3. The scalar product of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \, . \, \mathbf{B}$. The vector product is $\mathbf{A} \wedge \mathbf{B}$.

Dots ($\dot{a}$) denote differentiation with respect to time. Asterisks (*) denote the conjugate complex quantity. Bars ($^-$) are used for the coordinates of mean position. Primes ($'$) refer to points in the domain of integration (§ 6), or denote differentiation with respect to x (§ 5).

§1. INTRODUCTION: LIGHT CONSIDERED AS QUANTA AND ELECTRONS CONSIDERED AS WAVES

THE two primary elements of atomic physics—the quantum and the electron—possess a remarkable similarity in the properties with which they are credited. Indeed Beck* has even suggested that with every property of the electron there should be correlated some property of the quantum, so that, for example, we might associate the spin of an electron and the circular polarisation of a quantum. But the most arresting feature of this analogy is the absence of any compelling evidence for the existence either of free quanta or of free electrons.

* G. Beck, "Über einige Folgerungen aus dem Satz von der Analogie zwischen Lichtquant und Electron," *Zeits. f. Physik*, **43**, 658–74 (1927).

Modern quantum theory* is content with the quantisation of the optical energy transferred in processes of emission and absorption, and contemplates with equanimity the possibility that free radiation may be governed only by classical laws of continuity. It will be shown in this paper that a similar self-denying ordinance may be welcomed in the theory of electrons.

In the first place, it must be noted that any direct experiments, such as Millikan's, which purport to measure the electronic charge, are, in reality, competent to determine only the magnitudes of the electrical charges which can be absorbed or emitted by atoms. These experiments can establish at most that a quantity of negative electricity associated with an atom is a multiple of the unit which we call the "electronic charge": they necessarily fail to provide information regarding the atomicity or continuity of free electricity. It is true that such experiments as those conducted by Richardson and Brown appear to determine the distribution of velocity among free electrons and hence imply the atomicity of free electricity, but these experiments can also be interpreted without postulating the existence of free electrons.

In the second place, it is significant that Schrödinger's wave mechanics, which obliterates the discrete structure of a system of free electrons, preserves most carefully the integral character of the number of electronic charges bound to an atom. Since this theory does not require free electricity to be atomic in structure, an immediate exercise of Occam's razor frees us from the burden of such an assumption.

The old quantum theory replaced waves of light by quanta of optical energy. The new wave mechanics replaces quanta of electrical charge by waves of electricity, and replaces an atmosphere of free electrons by a spectral band of waves. In fact the fundamental equations of wave mechanics may be developed in such a way as to exhibit the transition from electrons to waves of electricity as a process which is reciprocal to the transition from waves of light to quanta with which we have been familiarised by the older quantum theory. That theory replaced a plane wave of light of frequency ν and wave-length λ by a stream of quanta each of energy $W = h\nu$ and of momentum $p = h/\lambda$, h being Planck's constant. In a similar manner the new wave mechanics replaces a uniform stream of electrons each of energy W and of momentum p by a plane wave of electricity of frequency $\nu = W/h$ and of wave-length $\lambda = h/p$.

A wealth of experimental evidence† is even now being accumulated in support of this theory. It has been shown that "a stream of electrons" may be scattered by amorphous films, diffracted by crystalline films or ruled gratings, and reflected from crystalline faces just as if it were a pencil of very hard X-rays. Moreover there is a fair numerical agreement between the experimental results and the predictions based upon the value of the wave-length given by the formula $\lambda = h/p$.

The analytical development of wave mechanics will take one of two parallel

* A. S. Eddington, *The Internal Constitution of the Stars*, § 41 (1926).

† B. L. Worsnop, "Note on the Transmission of Cathode Rays through Thin Films," *Proc. Phys. Soc.* **40**, 281–4 (1928).

routes according to the expression of the momentum p and energy W in the Newtonian form mv and $\frac{1}{2}mv^2$, or in the Einsteinian form $mv\ (1 - v^2/c^2)^{-\frac{1}{2}}$ and $mc^2\ (1 - v^2/c^2)^{-\frac{1}{2}}$, m being the rest-mass and v the velocity of the electron. The two values obtained for the wave velocity $\lambda\nu = W/p$ are respectively $\frac{1}{2}v$ and c^2/v. In spite of the inadmissible character of the first value some consideration will be devoted to "classical" wave mechanics for the following reasons:

1. The differential equations of this theory are easier to formulate, interpret and solve.

2. They provide a good first approximation to the equations of "relativistic" wave mechanics.

3. There is still a discussion as to the proper form to be taken by the equations of relativistic wave mechanics, e.g. Dirac and Darwin employ four complex wave formations and find difficulties in putting the wave equations in covariant form.

§2. THE FUNDAMENTAL EQUATIONS OF CLASSICAL WAVE MECHANICS

Since the wave theory of electricity receives experimental support from observations on free electricity it is worth while to investigate whether it can be harmonised with the old quantum theory of bound electricity which has been so successful in dealing with series spectra. But as soon as we endeavour to picture an atom in a stationary state as a positive nucleus surrounded by stationary waves of electricity we are again confronted by all the old difficulties associated with radiation from the moving charges which we imagine to be carried by the waves. One way out of these difficulties has been indicated by Lorentz*, who has suggested that the bound electricity might be distributed continuously around the "electronic orbit," so that the motion of the electric charge would give rise to a steady current emitting no radiation.

Now we can easily imagine a system of waves whose charge is concentrated near an "electronic orbit," but it is difficult to imagine any system of waves, *all of the same character*, giving rise to a steady distribution of electric charge. This difficulty may be evaded by introducing two systems of waves superimposed on one another and with suitable relations between their phases and amplitudes. We should thus obtain a state of affairs similar to the regime prevailing in a plane wave of light when the joint effect of the electric and magnetic waves is to produce a uniform distribution of optical energy.

2·1. *Construction of the Wave Equation.*

To give an analytical form to these assumptions, consider first the simplest type of wave—plane monochromatic waves of uniform amplitude. Let the two wave functions be the real and imaginary parts of

$$\psi = A \exp 2\pi\iota\, \{(l_1 x + l_2 y + l_3 z)/\lambda - \nu t\}.$$

* H. A. Lorentz, *Lectures on Theoretical Physics*, **2**, 329–33 (1927).

This double wave system is to replace a stream of electrons, each having energy $W = h\nu$ and components of momentum

$$p_1 = l_1 h/\lambda,\ \ p_2 = l_2 h/\lambda,\ \ p_3 = l_3 h/\lambda.$$

Hence, if $\mathbf{p}$ is the vector representing the momentum, and ∇ the vector operator with components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, then

$$(h/2\pi\iota)\,\nabla\psi = \mathbf{p}\psi \qquad \text{......(2·11)},$$

and

$$(-h/2\pi\iota)\,\partial\psi/\partial t = W\psi \qquad \text{......(2·12)}.$$

Since, for free electrons of mass m and velocity $\mathbf{v}$,

$$\mathbf{p} = m\mathbf{v},\ \ W = \tfrac{1}{2}mv^2,\ \text{and}\ \ p^2 = 2mW \qquad \text{......(2·13)},$$

therefore we find that the corresponding wave functions satisfy the equation

$$h.\nabla^2\psi + 4\pi m\iota.\partial\psi/\partial t = 0 \qquad \text{......(2·14)}.$$

Secondly, consider an electron moving under the influence of an electromagnetic field. Let V be the electrostatic potential and $\mathbf{A}$ the electromagnetic vector potential. If $-e$ is the electronic charge in electrostatic units we have the relations

$$\mathbf{p} = m\mathbf{v} - e\mathbf{A}/c,\ \ W = \tfrac{1}{2}mv^2 - eV \qquad \text{......(2·15)},$$

and

$$(\mathbf{p} + e\mathbf{A}/c)^2 = 2m\,(W + eV) \qquad \text{......(2·16)}.$$

It will accordingly be assumed that the equation for electric waves which interact with an electromagnetic field is

$$\frac{1}{2m}\left\{\frac{e}{c}\mathbf{A} + \frac{h}{2\pi\iota}\nabla\right\}^2.\psi = -\frac{h}{2\pi\iota}\frac{\partial\psi}{\partial t} + eV\psi \qquad \text{......(2·17)}.$$

2·2. *Lagrangian Form of the Wave Equation*†.

This equation (2·17), judged according to its analytical form, is not a wave equation but a diffusion equation, and in consequence it cannot be deduced from a Lagrangian function. But this difficulty can be evaded by considering only those wave functions which are simply periodic in time, so that, if ν is the frequency,

$$-\frac{h}{2\pi\iota}\frac{\partial\psi}{\partial t} = h\nu\psi.$$

If we write $\psi = \exp(2\pi\iota s/h)$, $\psi^* = \exp(-2\pi\iota s^*/h)$, using the asterisk to denote the conjugate complex quantity, and finally

$$L/\psi^*\psi = (h\nu + eV) - (e\mathbf{A}/c + \text{grad}\ s)\,(e\mathbf{A}/c + \text{grad}\ s^*)/(2m) \quad \text{...(2·21)},$$

it is easily verified that the wave equation (2·17) for wave functions with frequency ν is equivalent to the Lagrangian equation

$$\delta\int L\,dx\,dy\,dz = 0,$$

the variations being taken with respect to ψ^*.

† So much of the work is due to Schrödinger that only a general reference to his collected papers, now published in an English translation, can be given here.

To interpret this Lagrangian function, introduce the scalar ρ and the vector $\boldsymbol{\sigma}$ defined by the equations

$$\rho = -\,\partial L/\partial V = -\,e\psi^*\psi \qquad \text{......(2·22)},$$

$$\boldsymbol{\sigma} = +\,\partial L/\partial \mathbf{A} = (he\iota/4\pi mc)\,(\psi^*\,\text{grad}\,\psi - \psi\,\text{grad}\,\psi^*) - (e/mc^2)\,\mathbf{A}\rho \qquad \text{...(2·23)}.$$

If we agree that the vector potential is to satisfy the condition div $\mathbf{A} = 0$, it follows from the wave equation (2·17) that

$$c\,\text{div}\,\boldsymbol{\sigma} + \partial\rho/\partial t = 0 \qquad \text{......(2·24)},$$

so that ρ and $\boldsymbol{\sigma}$ satisfy the equation of continuity.

We may now write L in the form

$$\{h\nu\psi^*\psi - (h^2/8\pi^2 m)\,\text{grad}\,\psi\,.\,\text{grad}\,\psi^*\} + \mathbf{A}\,.\,\boldsymbol{\sigma} - V\rho + (e^2/2mc^2)\,A^2\psi^*\psi \qquad \text{...(2·25)}.$$

The Lagrangian function for the electromagnetic field is

$$\left.\begin{array}{lll} & L_0 = (E^2 - H^2)/8\pi, & \\ \text{where} & \mathbf{E} = -\,\text{grad}\,V - \partial\mathbf{A}/c\partial t & (\text{E.S.U.}) \\ \text{and} & \mathbf{H} = \text{curl}\,\mathbf{A} & (\text{E.M.U.}) \end{array}\right\} \qquad \text{......(2·26)}.$$

The interaction of the electromagnetic field and the electric waves may be deduced from the complete Lagrangian function $L_0 - L$. Variation with respect to ψ^* yields, as before, the wave equation (2·17). Variation with respect to V and $\mathbf{A}$ yields Maxwell's equations

$$\text{div}\,\mathbf{E} = 4\pi\rho, \quad \text{curl}\,\mathbf{H} - \partial\mathbf{E}/c\partial t = 4\pi\boldsymbol{\sigma} \qquad \text{......(2·27)}.$$

For this reason ρ is taken to be the volume density of the charge carried by the electric waves, while $\boldsymbol{\sigma}$ is taken to be the current density (E.M.U.).

Returning to the expression for the Lagrangian function (2·25) we note that the usual terms expressing the interaction of the charge and field, namely $(\mathbf{A}\,.\,\boldsymbol{\sigma} - V\rho)$, are supplemented by an additional small term $(e^2/2mc^2)\,A^2\psi^*\psi$, while the remaining terms, $h\nu\psi^*\psi$ and $(h^2/8\pi^2 m)\,\text{grad}\,\psi\,.\,\text{grad}\,\psi^*$, are most easily interpreted as the volume densities of the kinetic and potential energy carried by the electric wave.

§3. CLASSICAL ELECTRON THEORY AS A FIRST APPROXIMATION TO WAVE MECHANICS

The preceding section was devoted to the construction of the fundamental equations of classical wave mechanics. In this section it will be shown that, to a first approximation, the motion of electric charge as determined by the wave equation agrees with the predictions of the classical electron theory. The main theorems required are due to Madelung and Ehrenfest.

3·1. *Madelung's Hydrodynamical Form of the Wave Equation*†.

Since the wave function ψ is a complex quantity, we shall write

$$\psi = \alpha \exp\,(2\pi m\iota\beta/h) \qquad \text{......(3·11)},$$

where α and β are both real quantities. Then we find that

$$\rho = -\,e\alpha^2 \qquad \text{......(3·12)},$$

† E. Madelung, "Quantentheorie in hydrodynamischer Form," *Zeits. f. Physik*, **40**, 322–6 (1927); E. H. Kennard, "On the Quantum Mechanics of a System of Particles," *Phys. Rev.* **31**, 876–90 (1928).

and $$c\sigma = \rho\,(\text{grad}\,\beta + e\mathbf{A}/mc) = \rho\mathbf{q}, \text{ say} \qquad \text{......(3·13)}.$$

The wave equation for the complex function ψ yields two simultaneous equations for the real functions α and β, namely, the equation of conservation,

$$\text{div}\,(\rho\mathbf{q}) + \partial\rho/\partial t = 0 \qquad \text{......(3·14)},$$

and the "hydrodynamical equation,"

$$\partial\beta/\partial t + \tfrac{1}{2}q^2 - eV/m = (h^2\nabla^2\alpha)/(8\pi^2 m^2\alpha) \qquad \text{......(3·15)}.$$

This equation may be interpreted in two ways. First, let

$$U = (h^2/8\pi^2 m^2)\,(\nabla^2\alpha/\alpha) \quad (\text{cm.}^2/\text{sec.}^2) \qquad \text{......(3·16)}.$$

Then by taking the gradient of equation (3·15) we obtain the result

$$m\,\{\dot{\mathbf{q}} + (\mathbf{q}.\nabla)\,\mathbf{q}\} = -\,e\mathbf{E} + e\,[\mathbf{H}\wedge\mathbf{u}/c] + m\,\text{grad}\,U \quad \text{......(3·17)}.$$

Hence the electric charge carried by the waves moves as if it were carried by a perfect liquid in which the ratio of the electric density to the mass density is $(-\,e/m)$, the usual electric and magnetic forces being supplemented by a "quantum force," of magnitude

$$\mathbf{F} = -\,(\rho m/e)\,\text{grad}\,U, \text{ per unit volume.}$$

Secondly, writing the equation (3·15) in the Hamiltonian form

$$\partial\beta/\partial t + \tfrac{1}{2}\,\{\text{grad}\,\beta + e\mathbf{A}/mc\}^2 = U + eV/m \qquad \text{......(3·18)},$$

we see that $m\beta$ is the "action function" for an electron subjected to a "quantum force," $-\,m$ grad U, in addition to the usual electromagnetic forces. It follows that $\mathbf{q}$ would be the velocity of the electron at any point. Hence a system of electrons under the action of the usual electromagnetic forces plus the "quantum force" would move along the paths of flow of the current carried by the associated electric wave.

3·2. *Ehrenfest's Form of the Wave Equation**.

Let us consider solutions of the wave equation for which there exists a closed surface S over which the normal component of the current velocity $\sigma/\rho = \mathbf{q}/c$ is everywhere zero. The total quantity of electricity within this surface is

$$Q = \int \rho\, d\tau,$$

$d\tau$ being an element of volume, and on the basis of Madelung's theorem we shall call

$$M = (-\,m/e)\,Q,$$

the total "mass" associated with the waves inside S. The position vector of the "mass-centre" is defined by the equations

$$Q\,(\bar{x}, \bar{y}, \bar{z}) = \int (x, y, z)\,\rho\, d\tau.$$

We proceed to find expressions for the total momentum, $M d\bar{x}/dt$, ..., and the total effective force, $M d^2\bar{x}/dt^2$, ..., associated with the waves inside S.

* P. Ehrenfest, *Zeit. f. Physik*, **45**, 455–7 (1927); A. E. Ruark, "...The Force Equation and the Virial Theorem in the New Mechanics," *Phys. Rev.* **31**, 533–8 (1928).

Since $$\operatorname{div}(x\boldsymbol{\sigma}) = x \operatorname{div} \boldsymbol{\sigma} + \sigma_1{}^*, \qquad \text{......(3·21)},$$
therefore $$\left.\begin{aligned} Q d\bar{x}/dt &= \int x \dot{\rho}\, d\tau = -c\int x \operatorname{div} \boldsymbol{\sigma} . d\tau \\ &= c\int \sigma_1 d\tau - c\int x (\boldsymbol{\sigma} . d\mathrm{S}) \end{aligned}\right\} \qquad \text{......(3·22)},$$
the last integral being taken over the surface S. Since the normal component of $\boldsymbol{\sigma}$ vanishes over S, the total momentum
$$M d\bar{x}/dt = -(mc/e)\int \sigma_1 d\tau \qquad \text{......(3·23)}.$$

To calculate the total effective force consider the x-component of the vector $\rho\mathbf{E} + (\boldsymbol{\sigma} \wedge \mathbf{H})$. This is equal to
$$-\rho\frac{\partial V}{\partial x} - \frac{\rho}{c}\frac{\partial A_1}{\partial t} + \sigma_2\left\{\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right\} - \sigma_3\left\{\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right\}.$$
In this expression substitute
$$\mathbf{A} = (mc^2/e\rho)\,\boldsymbol{\sigma} - (mc/e) \operatorname{grad} \beta,$$
$$V = (m/e)\,\partial\beta/\partial t + (mc^2\sigma^2/2e\rho^2) - (m/e)\,U,$$
from equations (3·13) and (3·15). We thus obtain
$$-(mc/e)\,\dot{\sigma}_1 + (\rho m/e)\,\partial U/\partial x - (mc^2/e)\,I,$$
where $$I = \frac{\partial}{\partial x}\left\{\frac{\sigma_1{}^2}{\rho}\right\} + \frac{\partial}{\partial y}\left\{\frac{\sigma_1\sigma_2}{\rho}\right\} + \frac{\partial}{\partial z}\left\{\frac{\sigma_1\sigma_3}{\rho}\right\}.$$
The integral of I over the volume enclosed by S is evidently zero by Green's Theorem and the definition of S. Hence
$$M d^2\bar{x}/dt^2 = -(mc/e)\int\dot{\sigma}_1 d\tau = \int\{\rho E_1 + (\boldsymbol{\sigma} \wedge \mathbf{H})_1 - (\rho m/e)\,\partial U/\partial x\}\, d\tau \quad \text{...(3·24)}.$$
Thus the effective force on the mass-centre is the sum of the electromagnetic forces and the "quantum force" acting on the charge and current carried by the wave inside S. Hence the bulk motion of waves concentrated in small regions of space would be the same as the motion of an electron subjected to the same forces.

It only remains to note that the quantum force will only be of sensible magnitude on the boundaries of a beam where α is varying rapidly.

§4. ILLUSTRATIONS OF THE DISTORTION OF ELECTRIC WAVES BY ELECTROMAGNETIC FIELDS

In the preceding section the equivalence of the classical electron theory and of classical wave mechanics was established in a general way. This general theorem will now be illustrated by a number of particular examples drawn from the work of Schrödinger and Darwin and presented here in a simplified form based on Madelung's theorem.

4·1. *Uniform Electrostatic Field*†.

Let the potential of the field be $V = -Fx$. Assume provisionally that the wave carries a charge of uniform density $\rho = -e\alpha^2$, and a current with components of velocity
$$q_1 = \partial\beta/\partial x = u - eFt/m \qquad \text{......(4·11)},$$
$$q_2 = \partial\beta/\partial y = v \qquad \text{......(4·12)}$$

* $\sigma_1, \sigma_2, \sigma_3$ are the components of $\boldsymbol{\sigma}$.

† C. G. Darwin, "Free Motion in the Wave Mechanics," *Proc. R. S.* A, **117**, 258–93 (1927).

(see equations (3·12) and (3·13)), i.e. the same values as would be acquired by an electron projected initially with components u and v.

From the Hamiltonian form of the wave-equation (3·18) we find that

$$\partial\beta/\partial t = -eFx/m - \tfrac{1}{2}\{u^2 + v^2 - 2eFut/m + e^2F^2t^2/m^2\} \quad \text{......(4·13).}$$

These three equations ((4·11), (4·12) and (4·13)) are consistent and possess the solution

$$\beta = u(x - \tfrac{1}{2}ut) - eFxt/m + v(y - \tfrac{1}{2}vt) + eFut^2/2m - e^2F^2t^3/6m^2 \quad \text{...(4·14).}$$

Hence the provisional assumptions made above are justified.

4·2. *Linear Oscillator**.

The wave which corresponds to an electron moving in a field of potential $V = -2\pi^2\nu^2 m x^2/e$ will now be investigated. At time t, the electron would have a displacement $x = a\cos 2\pi\nu t$. Accordingly it will be assumed that the wave is concentrated around the point which would be occupied by the electron, and that its amplitude is

$$\alpha = \exp\frac{(x - a\cos 2\pi\nu t)^2}{-2\kappa^2} \quad \text{......(4·21).}$$

The Hamiltonian equation (3·18) now becomes

$$\partial\beta/\partial t + \tfrac{1}{2}(\partial\beta/\partial x)^2 = x^2\{(h^2/8\pi^2m^2\kappa^4) - 2\pi^2\nu^2\}$$
$$- (h^2/8\pi^2m^2\kappa^4)\{\kappa^2 - a^2\cos^2 2\pi\nu t + 2ax\cos 2\pi\nu t\} \quad \text{......(4·22).}$$

Adjusting κ so that the first term vanishes we find that a solution is

$$\left.\begin{aligned} \beta &= -2\pi\nu a x \sin 2\pi\nu t + f(t) \\ \text{where}\quad f(t) &= -h\nu t/2m + \tfrac{1}{2}\pi\nu a^2 \sin 4\pi\nu t \end{aligned}\right\} \quad \text{......(4·23).}$$

We still have to show that these expressions for α and β yield consistent values of the charge and current. From equation (3·13) we find that

$$c\sigma_1 \doteq -e\alpha^2 . \partial\beta/\partial x$$
$$= 2\pi\nu a e\alpha^2 \sin 2\pi\nu t \quad \text{......(4·24),}$$

whence

$$c \operatorname{div} \boldsymbol{\sigma} = 2e\alpha\dot{\alpha} = -\dot{\rho},$$

which is the condition of continuity required. Thus it has been shown that, corresponding to an electron vibrating in a linear oscillator, there exists a wave concentrated in a region around the position of the electron with dimensions of the order of

$$\kappa = \sqrt{(h/4\pi^2\nu m)}\ \text{cms.}$$

4·3. *Uniform Magnetic Field†*.

A uniform magnetic field of strength H directed along the z-axis has the vector potential whose components in cylindrical coordinates (z, R, ϕ) are

$$A_1 = 0,\ A_2 = 0,\ A_3 = \tfrac{1}{2}HR.$$

The Hamiltonian wave equation is

$$\partial\beta/\partial t + \tfrac{1}{2}(\partial\beta/\partial R)^2 + \tfrac{1}{2}(\partial\beta/R\partial\phi + \omega R)^2 = h^2\nabla^2\alpha/8\pi^2m^2\alpha \quad \text{......(4·31),}$$

where

$$\omega = eH/2mc \quad \text{......(4·32).}$$

* E. Schrödinger, "Der stetige Übergang von der Mikro- zur Makro-mechanik," *Die Naturwissenschaften*, 28, 664–6 (1926).

† *Loc. cit. sup.*

An electron in the same field would describe a circle $R = a$ with angular velocity 2ω. Hence we shall assume provisionally that the corresponding wave is concentrated around this moving point with amplitude

$$\alpha = \exp \frac{\{R^2 - 2Ra\cos(\phi - 2\omega t) + a^2\}}{-2\kappa^2} \qquad \text{......(4·33)}.$$

It follows that $\nabla^2\alpha/\alpha = -2/\kappa^2 + \{R^2 - 2Ra\cos(\phi - 2\omega t) + a^2\}/\kappa^4$.

Adjusting the value of κ so that the terms in R^2 disappear from the Hamiltonian equation (4·31), we find that a solution is

$$\beta = -(h\omega t/2\pi m) + a\omega R\sin(\phi - 2\omega t), \quad 2\pi\kappa^2 = h/m\omega \qquad \text{......(4·34)}.$$

From equation (3·13) it follows that the current carried by the wave has velocity components

$$q_1 = 0, \quad q_2 = \partial\beta/\partial R = a\omega\sin(\phi - 2\omega t)$$

and

$$q_3 = \partial\beta/R\partial\phi + eA_3/mc = a\omega\cos(\phi - 2\omega t) + \omega R \qquad \text{......(4·35)}.$$

It is easily verified that the components of the current density ρq_1, ρq_2, ρq_3 satisfy the equation of continuity and hence the consistency of our provisional solution is demonstrated.

§5. THE STATIONARY STATES OF AN ATOM

In § 2 the introduction of a complex wave function $\psi = \xi + i\eta$ was briefly justified on the ground that the joint action of both the real wave functions ξ and η would be required to produce a non-radiating distribution of electric charge in an atom. In this section the wave functions associated with an atom in a stationary state will be examined in detail. The nuclear charge will be supposed to be e and the electrostatic potential due to this charge to be e/r. For waves of frequency ν, corresponding to electrons of total energy $W = h\nu$, the wave equation (2·17) is

$$\nabla^2\psi + (8\pi^2 m/h^2)(W + e^2/r)\psi = 0 \qquad \text{......(5·01)}.$$

Since the energy of an electron in a closed orbit would be negative, we write

$$\left.\begin{aligned} h\kappa &= \sqrt{(-8\pi^2 mW)} \\ \text{and} \quad h^2\gamma &= 4\pi^2 me^2 \end{aligned}\right\} \qquad \text{......(5·02)},$$

so that the wave equation becomes

$$\nabla^2\psi = (\kappa^2 - 2\gamma r^{-1})\psi \qquad \text{......(5·03)}.$$

5·1. *The Existence of Discrete Energy Levels.*

It will now be shown that the whole of the waves associated with an atom in a stationary state has the same frequency and that the possible frequencies in different stationary states form a discrete series.

Let κ_1 and κ_2 be two possible values of κ corresponding to the possible values of the frequency, and let ψ_1 and ψ_2 be corresponding solutions of the wave equation (5·03). Then it follows that

$$\psi_2^*\nabla^2\psi_1 - \psi_1\nabla^2\psi_2^* = (\kappa_1^2 - \kappa_2^2)\psi_2^*\psi_1.$$

Integrate throughout a sphere of radius r. Then, by Green's Theorem,

$$(\kappa_1^2 - \kappa_2^2)\int\psi_2^*\psi_1 . d\tau = \int(\psi_2^* \operatorname{grad}\psi_1 - \psi_1 \operatorname{grad}\psi_2^* . dS),$$

$d\tau$ being a volume element and dS a surface element. If we agree that the wave functions are to behave like r^{-1} as r tends to infinity, we find that the volume integral of $\psi_2{}^*\psi_1$ throughout all space is zero unless $\kappa_1 = \kappa_2$ when it is obviously positive.

Now suppose that the possible range of frequencies to be found in stationary states includes a band extending from ν_1 to N, and that the corresponding range of values of κ extends from κ_1 to K. Consider the volume integral

$$\int_\infty \psi_2{}^*\psi_1 d\tau = f(\kappa_2)$$

as a function of κ_2. When $K > \kappa_2 > \kappa_1$, $f(\kappa_2) = 0$; when $\kappa_2 = \kappa_1$, $f(\kappa_2) > 0$; i.e. there is a discontinuity at $\kappa_2 = \kappa_1$ although $\psi_2{}^*$ and the volume integral are continuous functions of κ_2. Hence the possible range of frequencies cannot include a band, but must form a discrete series; and the same holds of the possible range of κ or W.

Let ν_1 and ν_2 be two terms of this series and let $u_1 \exp(-\omega_1 t)$ and $u_2 \exp(-\omega_2 t)$, ($\omega = 2\pi\iota\nu$), be corresponding solutions of (5·03), u_1 and u_2 being independent of t. Suppose that waves of both frequencies could be present in a stationary state. Then the complete wave function would be of the form

$$\psi = u_1 \exp(-\omega_1 t) + u_2 \exp(-\omega_2 t),$$

and the density of the charge carried by the waves would be

$$\rho = -e\psi^*\psi = -e(u_1{}^*u_1 + u_2{}^*u_2) - eu_2{}^*u_1 \exp(\omega_2 - \omega_1)t - eu_1{}^*u_2 \exp(\omega_1 - \omega_2)t.$$

Hence upon the static charge of density $-e(u_1{}^*u_1 + u_2{}^*u_2)$ would be superimposed a charge whose density fluctuates with frequency $|\nu_2 - \nu_1|$. This fluctuation would give rise to radiation of the same frequency, the electromagnetic field of the radiation would react on the electric wave system and the stationary state would be destroyed. Hence to preserve an atom in a stationary state all the waves must have the same frequency.

Thus it has been shown that the quantisation of the energy is a direct consequence of the form of the equation satisfied by the wave function ψ.

5·2. *The Values of the Discrete Energy Levels.*

In order to determine explicitly the permissible values of the frequency we note that the wave equation (5·03), when written in spherical polar coordinates (r, θ, ϕ), possesses elementary solutions of the form

$$u(\lambda, \mu) = \exp(-\kappa r + \lambda \log r \pm \iota\mu\phi) . f(2\kappa r) . P_\lambda^\mu(\cos\theta) \quad \text{......(5·21)},$$

where $f(x)$ satisfies the equation

$$xf'' + (2\lambda + 2 - x)f' - (\lambda + 1 - \gamma/\kappa)f = 0 \quad \text{......(5·22)}.$$

This equation possesses only one solution which is valid in the vicinity of the origin, $x = 0$, and which takes the value unity at that point. This solution may be represented by the series

$$f(x) = 1 + a_1 x + a_2 x^2/2! + \ldots + a_n x^n/n! + \ldots \quad \text{......(5·23)}$$

in which the coefficients are connected by the recurrence relation

$$(n + 2\lambda + 2)a_{n+1} = (n + \lambda + 1 - \gamma/\kappa)a_n \quad \text{......(5·24)}.$$

Now the only wave functions which are physically admissible are those which are single-valued and finite throughout all space. A reference to equation (5·21) shows that to satisfy these conditions μ must be a positive integer or zero, while λ must be a positive integer or zero and not less than μ. It will now be shown that there is also a limitation on the possible values of γ/κ.

If $(\gamma/\kappa - \lambda)$ is a positive integer n, the series (5·23) will terminate with the term $a_n x^n/n!$. But if $(\gamma/\kappa - \lambda)$ is not a positive integer, the series (5·23) will not terminate. Since the ratio

$$\frac{a_{n+1}}{a_n} = \frac{n + 2\lambda + 2}{n + \lambda + 1 - \gamma/\kappa}$$

tends to unity as $n \to \infty$, the series converges uniformly for all finite values of x and behaves like $\exp x$ as $x \to \infty$. Hence, as $r \to \infty$, the wave function $u(\lambda, \mu)$ behaves like $\exp(\kappa r + \lambda \log r)$ and therefore increases without limit. Therefore $(\gamma/\kappa - \lambda)$ must be a positive integer, n.

Under these circumstances

$$- W = - h\nu = h^2\kappa^2/8\pi^2 m = \frac{h^2\gamma^2}{8\pi^2 m(\lambda + n)^2} = \frac{2\pi^2 m e^4}{h^2(\lambda + n)^2} \quad \text{......(5·25)}$$

and $f(x)$ is a multiple of Sonine's polynomial $T_{2\lambda+1}^{n-1}(x)$, which is given by the series

$$T_m{}^n(x) = \frac{x^n}{(m+n)!\,n!} - \frac{x^{n-1}}{(m+n-1)!\,(n-1)!} + \dots$$
$$+ \frac{(-)^{p+n}x^p}{(m+p)!\,(n-p)!\,p!} - \dots + \frac{(-)^n}{m!\,n!} \quad \text{......(5·26).}$$

Since the value of the Rydberg constant is

$$N = 2\pi^2 m e^4/h^3,$$

the permissible values of the frequency are given by the equation

$$- \nu = N/(\lambda + n)^2 \quad \text{......(5·27).}$$

5·3. *Atomic Dimensions.*

That part of the wave function $u(\lambda, \mu)$, equation (5·21), which depends on r may be written as

$$\exp(-\kappa r + \lambda \log r)\,.\,T_{2\lambda+1}^{n-1}(2\kappa r).$$

When Sonine's polynomial is expanded in powers of $(2\kappa r)$ the wave function consists of the sum of a number of terms of the form

$$A \exp(-\kappa r)\,.\,r^{\lambda+p}, \qquad (p = 0, 1, 2, \dots, n-1).$$

Such a term reaches a maximum value when $\kappa r = \lambda + p$, and then decreases rapidly as r increases. Hence, when

$$r > (\lambda + n)/\kappa = (\lambda + n)^2/\gamma,$$

all the terms are decreasing in magnitude.

In Bohr's theory the radius of an orbit of energy $(-h\nu)$ was $e^2/(-2h\nu)$. But from equations (5·02) and (5·25)

$$- 2h\nu = \gamma e^2(\lambda + n)^{-2}$$

on the present theory. Hence the radius of Bohr's orbit is $(\lambda + n)^2 \gamma$. Therefore we see that outside a sphere of this radius the magnitude of the wave function rapidly diminishes. Hence the charge and current carried by the waves are confined to sphere with a radius of the order of magnitude of the corresponding Bohr orbit.

5·4. *The Bohr Magneton**.

Consider those stationary states of the atom for which the frequency of oscillation has the greatest possible values. If we take $\lambda + n = 1, 2$, the corresponding wave functions are those given below.

$\underline{\lambda + n = 1}$; i.e. $n = 1, \lambda = 0, \mu = 0,$

$$u = \exp(-r/\gamma):$$

$\underline{\lambda + n = 2}$; i.e. $n = 2, \lambda = 0, \mu = 0,$

$$u = (1 - 2r/\gamma)\exp(-2r/\gamma),$$

or $n = 1, \lambda = 1, \mu = 0,$

$$u = r\cos\theta . \exp(-2r/\gamma),$$

or $n = 1, \lambda = 1, \mu = \pm 1,$

$$u = r\sin\theta . \exp(-2r/\gamma \pm i\phi).$$

In the first four cases the wave carries a static charge but no current. In the last case the two wave functions may be written as

$$\psi_1 = f(r, \theta) . \exp(i\phi - 2\pi\iota\nu t),$$

$$\psi_2 = f(r, \theta) . \exp(-i\phi - 2\pi\iota\nu t).$$

Hence the most general wave function involving ϕ for this energy level is

$$\psi = f(r, \theta)\exp(-2\pi\iota\nu t)\{c_1 \exp\iota(\phi + \alpha_1) + c_2 \exp - \iota(\phi + \alpha_2)\},$$

where $c_1, c_2, \alpha_1, \alpha_2$ are real.

If the total charge carried by the wave is $-e$, it follows from equation (2·81) that

$$2\pi(c_1^2 + c_2^2)\int_0^\infty r^2 dr \int_0^\pi f^2 \sin\theta\, d\theta = 1.$$

The current has the direction of the ϕ-coordinate, i.e. it flows in circles around the z-axis. The current density is, from equation (2·23),

$$-(he/2\pi mc)(c_1^2 + c_2^2) f^2/(r\sin\theta).$$

The total magnetic moment due to a distribution of current with density $\boldsymbol{\sigma}$ at the point whose position vector is $\mathbf{r}$ is

$$\mathbf{M} = \tfrac{1}{2}\int[\mathbf{r}_\wedge \boldsymbol{\sigma}]\, d\tau.$$

Hence in this case it follows that the magnetic moment is along the z-axis and of magnitude

$$(he/2mc)(c_1^2 + c_2^2)\int_0^\infty r^2 dr \int_0^\pi f^2 \sin\theta\, d\theta = he/4\pi mc$$

$$= 1 \text{ Bohr magneton}.$$

Therefore the quantisation of the magnetic moment is a consequence of making the total charge of the waves equal to $-e$.

* E. Fermi, "Quantum Mechanics and the Magnetic Moment of Atoms," *Nature*, 118, 876 (1926); V. Fock, "Zur Schrödingerscher Wellenmechanik," *Zeit. f. Physik*, 38, 242–50 (1926).

5·5. *The Zeeman Effect.*

Let us determine the change in the characteristic frequencies produced by the application of a magnetic field of uniform strength H along the z-axis. As in § 4, the components of the vector potential are

$$A_1 = 0, \; A_2 = 0, \; A_3 = \tfrac{1}{2}HR,$$

in cylindrical coordinates (z, R, ϕ). It follows from equation (2·17) that the wave equation is

$$\nabla^2\psi + \frac{2\pi\iota eH}{hc}\frac{\partial\psi}{R\partial\phi} = -\frac{4\pi m\iota}{h}\frac{\partial\psi}{\partial t} - \frac{2\gamma}{r}\psi,$$

neglecting squares of H. Put $\phi = \phi_1 + \omega t,$

where $\omega = eH/2mc,$

and this equation becomes

$$\nabla_1{}^2\psi = -\frac{4\pi m\iota}{h}\frac{\partial\psi}{\partial t} - \frac{2\gamma}{r}\psi \qquad \text{......(5·51)},$$

where $$\nabla_1{}^2\psi \equiv \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial R^2} + \frac{\partial\psi}{R\partial R} + \frac{\partial^2\psi}{R^2\partial\phi_1{}^2}.$$

Equation (5·51) has the same form as the wave equation (5·03) in the absence of a magnetic field. Hence the wave function has the form

$$\exp(\pm\,\iota\mu\phi_1 - 2\pi\iota\nu t).f(r,\theta) = \exp(\pm\,\iota\mu\phi \pm \iota\mu\omega t - 2\pi\iota\nu t).f(r,\theta),$$

and the frequencies are now given by the expression

$$\nu \pm \mu\omega/2\pi = (N/n^2) \pm (\mu\omega/2\pi),$$

where μ is, of course, either an integer or zero.

§6. THE EMISSION OF RADIATION

In this section the transitions by which an atom passes from one energy level to another will be studied. The principal results which will be obtained are the following:

a. The classical "selection rules" of spectral theory.

b. The emission of radiation in quanta.

c. The intensities of spectral lines.

These results are all reached by working with a wave equation modified to allow for the reaction of the radiation on the atom.

Although there may be several stationary states associated with each energy level the discussion will be limited for the sake of simplicity to the simplest type of transition, namely transition from one stationary state to another.

The method employed is one of successive approximations. If

$$\psi_1 = u_1 \exp(-\,\omega_1 t) \text{ and } \psi_2 = u_2 \exp(-\,\omega_2 t)$$

are the wave functions in the initial and final states, we assume that during the process of transition the wave function has the form

$$\psi = a_1\psi_1 + a_2\psi_2,$$

where a_1 and a_2 are coefficients depending only on the time. Since the process of

transition must be slow compared with the frequency of oscillation of ψ_1 and ψ_2, we ignore the variation of a_1 and a_2 with the time in calculating the instantaneous values of the charge and current density carried by the wave. From these values we calculate the electromagnetic potentials of the field of radiation emitted and hence the rate of emission of energy. We then return to the wave equation and introduce a correcting term to allow for the reaction of the radiation on the waves. We can then determine the rates of charge of a_1 and a_2 and thus estimate the probability of a transition.

6·1. *The Electromagnetic Potentials.*

If we write and

$$\left.\begin{aligned}\rho_{kn} &= -eu_k^* u_n\\ \boldsymbol{\sigma}_{kn} &= (he\iota/4\pi mc)\,(u_k^* \operatorname{grad} u_n - u_n \operatorname{grad} u_k^*)\end{aligned}\right\} \qquad \text{......(6·11),}$$

the instantaneous values of the charge and current density are

$$\left.\begin{aligned}\rho &= \Sigma a_k^* a_n \rho_{kn} \exp \omega_{kn} t\\ \boldsymbol{\sigma} &= \Sigma a_k^* a_n \boldsymbol{\sigma}_{kn} \exp \omega_{kn} t\end{aligned}\right\} \qquad \text{......(6·12),}$$

where

$$\omega_{kn} = 2\pi\iota\,(\nu_k - \nu_n) \qquad \text{......(6·13).}$$

When we are considering the values of these quantities at the point whose position vector is $\mathbf{r}'$, this fact will be indicated by attaching primes to the corresponding symbols.

The values of the scalar and vector potentials due to the charge and current carried by the waves are calculated at the point whose position vector is $\mathbf{r}$, and they are given by the equations

$$V = \Sigma a_k^* a_n V_{kn}, \quad \mathbf{A} = \Sigma a_k^* a_n \mathbf{A}_{kn} \qquad \text{......(6·14),}$$

$$\left.\begin{aligned}V_{kn} &= \int \exp \omega_{kn}(t - R/c)\,.\,\rho'_{kn}/R\,.\,d\tau'\\ \mathbf{A}_{kn} &= \int \exp \omega_{kn}(t - R/c)\,.\,\boldsymbol{\sigma}'_{kn}/R\,.\,d\tau'\end{aligned}\right\} \qquad \text{......(6·15),}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| = r\{1 - 2(\mathbf{r}\,.\,\mathbf{r}')\,r^{-2} + (r'/r)^2\}^{\frac{1}{2}} \qquad \text{......(6·16).}$$

Since the "partial potentials" V_{kn} and $\mathbf{A}_{kn}$ are the only functions which involve the factor $\exp \omega_{kn} t$, they must separately satisfy the equation

$$c \operatorname{div} \mathbf{A}_{kn} + \partial V_{kn}/\partial t = 0.$$

Hence if we define the vector $\mathbf{s}_{kn}$ by the equation

$$\omega_{kn} \mathbf{s}_{kn} = c \int \exp \omega_{kn}(t - R/c)\,.\,\boldsymbol{\sigma}'_{kn}/R\,.\,d\tau' \qquad \text{......(6·17),}$$

we may write

$$\left.\begin{aligned}c\mathbf{A} = \partial \mathbf{s}/\partial t, \qquad V &= -\operatorname{div} \mathbf{s}\\ \mathbf{s} &= \Sigma a_k^* a_n \mathbf{s}_{kn}\end{aligned}\right\} \qquad \text{......(6·18).}$$

where

It may now be verified that the magnetic and electric intensities are

$$\mathbf{H} = c^{-1} \operatorname{curl} \dot{\mathbf{s}}, \quad \mathbf{E} = \operatorname{curl} \operatorname{curl} \mathbf{s} \qquad \text{......(6·19).}$$

6·2. *Dipole Radiation.*

It was shown in § 5·3 that the greater part of the charge and current are confined to the interior of a sphere of radius a_n with its centre at the nucleus, a_n being the radius of the corresponding orbit in Bohr's theory. Now

$$2h\nu_n = e^2/a_n \qquad \text{......(6·21).}$$

Hence $\quad | \omega_{kn} a_n / c | < 2\pi\nu_n a_n / c = \pi e^2 / hc = 0\cdot0036$ (dimensionless) $\quad$ (6·22),

and $|\omega_{kn}|.(\mathbf{r}\mathbf{r}')(rc)^{-1}$ is of the same order when $r' \doteqdot a_n$. Therefore if r is chosen to be large compared with a_n, we may replace $R^{-1} \exp \omega_{kn}(t - R/c)$ by $r^{-1} \exp \omega_{kn}(t - r/c)$ in equations (6·15) and (6·17), and limit the range of integration to the interior of the sphere $r' = 10a_n$ (say), thus obtaining

$$r\omega_{kn}\mathbf{s}_{kn} = c \exp \omega_{kn}(t - r/c).\int \boldsymbol{\sigma}'_{kn} d\tau' \qquad \text{......(6·23).}$$

It is really immaterial whether the range of integration is thus restricted or not. If we include all space in the range we may transform the integral as in equations (3·21), (3·22). In virtue of the wave-equation (5·03) and the definitions of (6·11) we find that

$$c \operatorname{div} \boldsymbol{\sigma}_{kn} + \omega_{kn}\rho_{kn} = 0 \qquad \text{......(6·24).}$$

Now define the polarisation $\mathbf{p}_{kn}$ by the integral

$$\mathbf{p}_{kn} = \int \mathbf{r}'\rho'_{kn} d\tau' = \mathbf{p}^*_{nk}, \qquad \text{......(6·25),}$$

then $\quad c\int \boldsymbol{\sigma}'_{kn} d\tau' = - c\int \mathbf{r}' \operatorname{div} \boldsymbol{\sigma}'_{kn} d\tau = \omega_{kn}\int \mathbf{r}'\rho'_{kn} d\tau' = \omega_{kn}\mathbf{p}_{kn} \quad$ ......(6·26).

Hence $\quad \mathbf{s}_{kn} = \mathbf{p}_{kn} r^{-1} \exp \omega_{kn}(t - r/c) \quad$ ......(6·27).

By referring to equation (6·18) it will now be seen that at great distances from the nucleus the field produced by the waves consists of a steady electrostatic field superimposed on an oscillating field of the type due to a Hertzian doublet or dipole of strength

$$\mathbf{p}_{12} \exp \omega_{12} t + \mathbf{p}_{21} \exp \omega_{21} t \qquad \text{......(6·28).}$$

6·3. *The Selection Rules.*

In spherical polar coordinates (r, θ, ϕ) we have

$$\mathbf{p}_{12} = - e \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \mathbf{r} u_1^* u_2 d\phi$$

(dropping the primes in the integral). Let

$$u_1 \exp(-2\pi\iota\nu_1 t) = \psi(\lambda_1, \mu_1, \nu_1),$$
$$u_2 \exp(-2\pi\iota\nu_2 t) = \psi(\lambda_2, \mu_2, \nu_2),$$

in the notation of equation (5·21).

The following results may easily be verified:

I. If $\mu_1 = \mu_2$, $\mathbf{p}_{12}$ is parallel to the z-axis.

II. If $\mu_1 - \mu_2 = \pm 1$, $\mathbf{p}_{12}$ is perpendicular to the z-axis.

III. If none of these relations is satisfied, $\mathbf{p}_{12} = 0$.

These results constitute the equatorial selection principle. By employing well-known results in harmonic analysis it may be shown that

IV. If $\lambda_1 - \lambda_2 \neq \pm 1$, $\mathbf{p}_{12} = 0$.

This is the azimuthal selection principle.

If the polarisation $\mathbf{p}_{12}$ vanishes, it is evident that the transition does not give rise to any radiation, for the strength of the equivalent Hertzian doublet is zero (6·28).

It also follows from the form of the integral for $\mathbf{p}_{12}$ that this quantity is wholly real. Henceforward let us write for the sake of brevity

$$\mathbf{p}_{12} = \mathbf{p} \text{ and } \omega = - i\omega_{12} = 2\pi(\nu_1 - \nu_2) \qquad \text{......(6·31).}$$

6·4. *The Rate of Emission of Energy.*

It follows from equations (6·19) and (6·27) that the magnetic intensity of the radiation field is
$$\mathbf{H} = (\omega/c)^2 \operatorname{grad} \mathbf{r} \wedge (a_1{}^* a_2 \mathbf{s}_{12} + a_2{}^* a_1 \mathbf{s}_{21}).$$
Hence if the axes be chosen so that the polarisation $\mathbf{p}$ is directed along the z-axis,
$$H^2 = \frac{\omega^4 p^2 \sin^2\theta}{r^2 c^4} \{a_1{}^* a_2 \exp \omega_{12} (t - r/c) + a_2{}^* a_1 \exp \omega_{21} (t - r/c)\}^2$$
$$= \frac{4\omega^4 a_1 a_2 a_1{}^* a_2{}^* p^2}{r^2 c^4} \sin^2\theta \,.\, \cos^2 \omega (t - r/c - \alpha),$$
where α is the argument of $a_1{}^* a_2$.

Since the radiation is due to a Hertzian doublet, the electric and magnetic intensities are equal in magnitude and perpendicular in direction when r is large. Therefore the rate at which energy crosses a large sphere of radius B is
$$\int_0^\pi (c/4\pi) [\mathbf{E} \wedge \mathbf{H}]_B \,.\, 2\pi B^2 \sin\theta \, d\theta$$
$$= \tfrac{1}{2} c \int_0^\pi H^2{}_B \,.\, B^2 \sin\theta \, d\theta$$
$$= \frac{8\omega^4 a_1 a_2 a_1{}^* a_2{}^* p^2}{3c^3} \cos^2 \omega (t - B/c - \alpha) \qquad \text{......(6·41)†}.$$

6·5. *The Modified Wave Equation.*

Fermi‡ modified the wave equation by introducing an extra term into the electrostatic potential V to represent the reaction of the radiation. But, since a field of radiation always requires the electromagnetic vector potential for its specification, it seems better to proceed as follows.

An electron moving with an acceleration $\boldsymbol{\alpha}$ radiates energy at the rate $2e^2\alpha^2/3c^3$ ergs per second and experiences a retarding force of $2e^2\dot{\boldsymbol{\alpha}}/3c^3$ dynes. This is the force which would arise from a vector potential
$$\mathbf{A} = (2e/3c^2) \,.\, \boldsymbol{\alpha}.$$
Now it has been shown (§ 6·4 that the atom radiates energy at the rate $2\omega^4\beta^2/3c^3$ where
$$\boldsymbol{\beta} = a_1{}^* a_2 \mathbf{p} \exp \omega_{12} (t - B/c) + a_2{}^* a_1 \mathbf{p} \exp \omega_{21} (t - B/c) \qquad \text{......(6·51)}.$$
Hence we shall modify the wave equation by introducing the vector potential
$$\mathbf{A} = (2\omega^2/3c^2) \, \boldsymbol{\beta} \qquad \text{......(6·52)}.$$
The wave equation now becomes
$$L(\psi) \equiv \frac{\partial\psi}{\partial t} + \frac{h}{4\pi m i} \nabla^2\psi - \frac{2\pi i e}{h} V_0 \psi + \frac{e}{mc} (\mathbf{A}_0 \,.\, \operatorname{grad} \psi) = -\frac{e}{mc} (\mathbf{A} \,.\, \operatorname{grad} \psi),$$
V_0 and $\mathbf{A}_0$ being potentials of the nucleus and any external field applied to the atom. If in this equation we substitute
$$\psi = a_1\psi_1 + a_2\psi_2,$$
where ψ_1 and ψ_2 satisfy $L(\psi) = 0$, we find that
$$\dot{a}_1\psi_1 + \dot{a}_2\psi_2 = -(e/mc) \,.\, (\mathbf{A} \,.\, a_1 \operatorname{grad} \psi_1 + a_2 \operatorname{grad} \psi_2) \qquad \text{......(6·53)}.$$

† M. Bohr and P. Jordan, "Zur Quantenmechanik," *Zeits. f. Physik*, **34**, 858–88 (1925).

‡ E. Fermi, "Sul meccanismo dell' emissione nel meccanismo ondulatorio," *Accad. Lincei* (6), **5**, 795–800 (1927).

6·6. *Dirac's Equations.*

We now obtain the differential equations for a_1 and a_2 in the form given by Dirac†. Multiply equation (6·53) by $\psi_k{}^*$ and integrate over all space. If we write

$$a_{kn} = -(e/mc)\int(\mathbf{A}.\psi_k{}^* \operatorname{grad}\psi_n)\,d\tau \qquad \text{......(6·61),}$$

we find that

$$\dot{a}_1\int\psi_k{}^*\psi_1 d\tau + \dot{a}_2\int\psi_k{}^*\psi_2 d\tau = a_{k1}a_1 + a_{k2}a_2 \qquad \text{......(6·62).}$$

Since the total charge carried by the waves remains constantly equal to $-e$, we deduce that

$$\int(a_1{}^*a_2\psi_1{}^*\psi_2 + a_1{}^*a_2\psi_1{}^*\psi_2 + a_2{}^*a_1\psi_2{}^*\psi_1 + a_2{}^*a_2\psi_2{}^*\psi_2)\,d\tau = 1.$$

If, for convenience, we assume the functions ψ_1, ψ_2, normalised so that

$$\int\psi_1{}^*\psi_2 d\tau = 1 = \int\psi_2{}^*\psi_2 d\tau,$$

we may conclude that

$$a_1{}^*a_1 + a_2{}^*a_2 = 1 \qquad \text{......(6·63),}$$

and

$$\int\psi_1{}^*\psi_2 d\tau = 0 = \int\psi_2{}^*\psi_1 d\tau.$$

Hence equations (6·62) become

$$\left.\begin{aligned}\dot{a}_1 &= a_{11}a_1 + a_{12}a_2\\ \dot{a}_2 &= a_{21}a_1 + a_{22}a_2\end{aligned}\right\} \qquad \text{......(6·64).}$$

These are Dirac's equations. Now from equation (6·61)

$$a^*{}_{nk} = -(e/mc)\int(\mathbf{A}.\psi_n \operatorname{grad}\psi_k{}^*)\,d\tau,$$

and

$$a_{kn} + a^*{}_{kn} = -(e/mc)\int\operatorname{div}(A\psi^*{}_k\psi_n)\,d\tau = 0,$$

since ψ_n contains a factor $\exp(-kr)$. Therefore

$$\begin{aligned}a_{kn} &= \tfrac{1}{2}(a_{kn} - a^*{}_{nk})\\ &= (2\pi\iota/h)\exp\omega_{kn}t\,.\int(\mathbf{A}.\boldsymbol{\sigma}_{kn})\,d\tau\\ &= (2\pi\iota\omega_{kn}/hc)\exp\omega_{kn}t\,(\mathbf{A}.\mathbf{p}_{kn})\end{aligned} \qquad \text{......(6·65),}$$

from (6·26).

6·7. *The Emission of Quanta.*

Since it is only the secular charges in a_1 and a_2 which will be revealed in experiment, only the constant coefficients on the right-hand side of equations (6·64) will be retained, the periodic terms being neglected. On introducing the values of a_{kn} and $\mathbf{A}$ from (6·65), (6·51) and (6·52), we obtain Fermi's equations

$$\dot{a}_1 = -Aa_1a_2{}^*a_2, \quad \dot{a}_2 = +Aa_1{}^*a_1a_2,$$

where

$$A = 4\pi\omega^3p^2/3hc^3 \qquad \text{......(6·71).}$$

Hence

$$d(a_1{}^*a_1)/dt = -2Aa_1a_2a_1{}^*a_2{}^* \qquad \text{......(6·72)}$$

and

$$\int_{-\infty}^{\infty} a_1a_2a_1{}^*a_2{}^*dt = -(2A)^{-1}\Big[a_1{}^*a_2\Big]_{-\infty}^{\infty}.$$

But in a transition $(\nu_1 \rightarrow \nu_2)$ the quantity $a_1{}^*a_1$, passes from unity to zero (6·63). Therefore

$$\int_{-\infty}^{\infty} a_1a_2a_1{}^*a_2{}^*dt = (2A)^{-1} \qquad \text{......(6·73).}$$

† P. A. M. Dirac, "On the Theory of Quantum Mechanics," *Proc. R. S. A*, 112, 661–77 (1926).

The total amount of energy radiated during the transition is

$$E_{12}=\frac{8\omega^4p^2}{3c^3}\int_{-\infty}^{\infty} a_1a_2a_1^*a_2^*\cos^2\omega\,(t-B/c-\alpha)\,dt,$$

from (6·41). Since the integral is the product of a rapidly fluctuating periodic term and a slowly varying term, we may replace the former by its mean value, which is one-half. We then find from (6·71) and (6·73) that

$$E_{12}=2\omega^4p^2/3c^3A=h\,(\nu_1-\nu_2) \qquad \text{......(6·74)}.$$

It follows from (6·71) and (6·31) that if

$$0>\nu_1>\nu_2,$$

A and ω are positive, and that the transition is accompanied by the emission of a quantum of radiation of frequency $\nu=\nu_1-\nu_2$.

6·8. *The Intensities of Spectral Lines.*

It follows from equations (6·63) and (6·72) that

$$a_1^*a_1=\{1+\exp 2A\,(t-t_0)\}^{-1},\quad a_2^*a_2=\exp 2A\,(t-t_0)\,\{1+\exp 2A\,(t-t_0)\}^{-1},$$

while

$$a_1^*a_1a_2^*a_2=\tfrac{1}{4}\operatorname{sech}^2 A\,(t-t_0),$$

t_0 being the time at which the transition is half completed. Hence from (6·41) the maximum mean rate of emission of energy is

$$\omega^4p^2/3c^3.$$

It is accordingly usual to assume that the intensity of the spectral line of frequency ν_{kn} emitted by the atom is proportional to $\nu^4_{kn}p^2_{kn}$ where p_{kn} is the polarisation for the transition $(\nu_k\to\nu_n)$.

If A_{kn} is the "probability" of a transition, the rate of emission of energy from M atoms would be $MA_{kn}h\nu_{kn}$, since one quantum is emitted in each transition. Therefore the coefficient of emission, A_{kn}, is proportional to $\nu^3_{kn}p^2_{kn}$.

§7. THE PRINCIPLES OF RELATIVISTIC WAVE MECHANICS

In this section the principles of relativistic wave mechanics of unpolarised waves will be developed briefly, the discussion following a course parallel to the investigations of §§ 2 and 3. By way of illustration a simple theory of the Compton effect will be added.

7·1. *Construction of the Wave Equation.*

Consider the motion of an electron, of charge $-e$ and proper mass m, moving with velocity $\mathbf{v}$ in an electromagnetic field with scalar and vector potentials V and $\mathbf{A}$. The Lagrangian function is

$$\Lambda=-mc^2(1-v^2/c^2)^{\frac{1}{2}}+eV-(\mathbf{A}.\mathbf{v})\,(e/c),$$

the momentum is

$$\mathbf{p}=\partial\Lambda/\partial\mathbf{v}=m\mathbf{v}\,(1-v^2/c^2)^{-\frac{1}{2}}-e\mathbf{A}/c,$$

and the total energy of the electron is

$$W=(\mathbf{p}.\mathbf{v})-\Lambda=mc^2(1-v^2/c^2)^{-\frac{1}{2}}-eV.$$

Hence the total energy and the components of momentum are connected by the relation (compare equation (2·16))

$$m^2c^2 + (\mathbf{p} + e\mathbf{A}/c)^2 = (W + eV)^2 c^{-2}.$$

As in § 2, we multiply this equation through by the wave function ψ and then replace $\mathbf{p}\psi$ by $(h/2\pi\iota)\,\nabla\psi$, and $W\psi$ by $(-h\partial\psi/2\pi\iota\partial t)$ (equations (2·11) and (2·12)), thus obtaining as one general wave equation

$$\left(\frac{e}{c}\mathbf{A} + \frac{h}{2\pi\iota}\nabla\right)^2 \psi + m^2c^2\psi = \left(eV - \frac{h}{2\pi\iota}\frac{\partial}{\partial t}\right)^2 \psi/c^2 \qquad \text{......(7·11).}$$

On making use of the relation

$$c \operatorname{div} \mathbf{A} + \partial V/\partial t = 0,$$

this reduces to

$$\begin{aligned}\nabla^2\psi - \partial^2\psi/c^2\partial t^2 &= (2\pi mc/h)^2\psi + (2\pi e/hc)^2 (A^2 - V^2)\psi \\ &\quad - (4\pi\iota e/hc)\{(\mathbf{A}.\operatorname{grad}\psi) + (V\partial\psi/c\partial t)\} \qquad \text{......(7·12).}\end{aligned}$$

The accurate relation between the wave-length $\lambda = h/p$, and the frequency $\nu = W/h$ for free electric waves is derived from the relation

$$m^2c^2 + p^2 = W^2/c^2,$$

which yields

$$\lambda^2 (\nu^2 - \nu_0^2) = c^2 \qquad \text{......(7·13),}$$

where $\nu_0 = mc^2/h$ is the "threshold frequency" which forms a natural lower limit to the possible frequencies of free electric waves.

7·2. *Lagrangian Form of the Wave Equation*†.

As in § 2 we write

$$\psi = \exp(2\pi\iota s/h) \text{ and } \psi^* = \exp(-2\pi\iota s^*/h),$$

and retain the same Lagrangian function, L_0 (2·26), for the electromagnetic field. We find that the wave equation (7·12) may be deduced from the Lagrangian function L, where

$$L/\psi^*\psi = m^2c^2 + (\operatorname{grad} s + e\mathbf{A}/c)(\operatorname{grad} s^* + e\mathbf{A}/c) - (eV - \dot{s})(eV - \dot{s}^*)\,c^{-2} \quad \text{(7·21).}$$

To interpret this function, let

$$\rho = -\,\partial L/\partial V = -\frac{he}{2\pi\iota c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) + \frac{2e^2}{c^2}\psi^*\psi V \qquad \text{......(7·22),}$$

and

$$\boldsymbol{\sigma} = \partial L/\partial\mathbf{A} = \frac{he}{2\pi\iota c}(\psi^*\operatorname{grad}\psi - \psi\operatorname{grad}\psi^*) + \frac{2e^2}{c^2}\psi^*\psi\mathbf{A} \qquad \text{......(7·23),}$$

then L may be rewritten as

$$\begin{aligned}m^2c^2\psi^*\psi + (h^2/4\pi^2)\left(\operatorname{grad}\psi.\operatorname{grad}\psi^* - \frac{\partial\psi}{c\partial t}\cdot\frac{\partial\psi^*}{c\partial t}\right) - (A^2 - V^2)\,e^2\psi^*\psi/c^2 \\ + (\mathbf{A}.\boldsymbol{\sigma}) - V\rho \qquad \text{......(7·24).}\end{aligned}$$

As in § 2, if we vary V and $\mathbf{A}$ in the complete Lagrangian function $L_0 + L$, we obtain Maxwell's equations

$$\operatorname{div}\mathbf{E} = 4\pi\rho \text{ and } \operatorname{curl}\mathbf{H} - \partial\mathbf{E}/c\partial t = 4\pi\boldsymbol{\sigma}.$$

† W. Gordon, "Der Comptoneffekt nach der Schrödingerschen Theorie," *Zeits. f. Physik*, **40**, 117–33 (1926).

Hence ρ is the volume density of the charge carried by the waves while $\boldsymbol{\sigma}$ is the current density; and from these relations the equation of continuity

$$c \operatorname{div} \boldsymbol{\sigma} + \partial\rho/\partial t = 0 \qquad \text{......(7·25)}$$

follows at once. It will now be evident which parts of the total action density L are to be attributed to the electric waves and which part to their interaction with the electromagnetic field.

7·3. *The Relativistic Form of Madelung's Equation.*

It will now be shown that to a first approximation the motion of electric charge determined by the wave equation (7·12) agrees with the predictions of the relativistic electron theory. As in § 3·1 we write

$$\psi = \alpha \exp(2\pi\iota\beta/h) \qquad \text{......(7·31),}$$

where α and β are both real quantities. Then we find from equations (7·22) and (7·23) that

$$\rho = -(2e\alpha^2 c^{-2})(\partial\beta/\partial t - eV) \qquad \text{......(7·32),}$$

and

$$\boldsymbol{\sigma} = (2e\alpha^2 c^{-1})(\operatorname{grad}\beta + e\mathbf{A}/c) \qquad \text{......(7·33).}$$

The wave equation is now equivalent to the equation of conservation (7·25) and the generalisation of Madelung's equation, namely

$$4e^2m^2 + (\sigma^2 - \rho^2)\alpha^{-4} = (h^2e^2/\pi^2c^2\alpha)(\nabla^2\alpha - \ddot{\alpha}/c^2) \qquad \text{......(7·34).}$$

If this equation be rewritten in the form

$$m^2c^2 + (\operatorname{grad}\beta + e\mathbf{A}/c)^2 - (\partial\beta/\partial t - eV)^2 c^{-2} = (h^2/4\pi^2\alpha)(\nabla^2\alpha - \ddot{\alpha}/c^2) \text{ ......(7·35),}$$

it will be recognised as the Hamiltonian equation for an electron whose internal energy is not mc^2 but

$$mc^2 - (h^2/4\pi^2 m\alpha)(\nabla^2\alpha - \ddot{\alpha}/c^2).$$

Since the additional term is in general small we see that β is, approximately, the "action-function" for an electron.

Hence, according to the principles of classical relativity, a system of electrons projected into an electromagnetic field of potentials V and $\mathbf{A}$ would move so that at any point their velocity $\mathbf{v}$ was given by the equations

$$m\mathbf{v}(1 - v^2/c^2)^{-\frac{1}{2}} = \operatorname{grad}\beta + e\mathbf{A}/c,$$

$$mc^2(1 - v^2/c^2)^{-\frac{1}{2}} = \partial\beta/\partial t - eV,$$

whence

$$\mathbf{v} = c\boldsymbol{\sigma}/\rho.$$

Therefore the velocity which the electrons would have at any point agrees with the velocity with which the electric charge is carried by the waves.

7·4. *The Compton Effect**.

Only the wave-length of the scattered light and not its intensity will be determined here. The method adopted may be outlined as follows: (*a*) An atmosphere of electrons under the action of an incident beam of light is considered. (*b*) The

* P. A. M. Dirac, "The Compton Effect in Wave Mechanics," *Proc. Camb. Phys. Soc.* **23**, 500–7 (1926); "The Relativity Quantum Mechanics with an Application to Compton Scattering," *Proc. R. S.* A, **111**, 405–23 (1926); W. Gordon, *loc. cit.*, *supra*; E. Schrödinger, "Über den Comptoneffekt," *Ann. d. Phys.* 82, 257–64 (1927).

complete wave function for a system of free electrons may be built up from elementary wave functions each of which is associated with electrons of the same energy and momentum. (*c*) The small changes which are produced in these elementary wave functions by the incident beam are determined, and then (*d*) the changes in the charge and current carried by the waves are calculated. (*e*) From these results the wave-length of the radiation emitted in any chosen direction can then be determined.

(*a*) Let the incident light have frequency ν_1, and wave-length λ_1, and let its direction of propagation be given by the unit vector $\mathbf{n}_1$. If we write

$$\left.\begin{aligned} E_1 = h\nu_1,\ \mathfrak{f}_1 = h\mathbf{n}_1/\lambda_1, \\ h\phi_1 = 2\pi\{(\mathfrak{f}_1 . \mathbf{r}) - E_1 t\} \end{aligned}\right\} \qquad \text{......(7·41)},$$

we may take the scalar and vector potentials of the incident beam to be

$$V = 0,\ \mathbf{A} = \mathbf{a}\cos\phi_1 \qquad \text{......(7·42)}.$$

(*b*) A system of electrons all of energy W_n and momentum $\mathbf{p}_n$ is replaced in our wave mechanics by a wave function

$$\psi_n = \alpha_n \exp(2\pi\iota/h)\{(\mathbf{p}_n . \mathbf{r}) - W_n t\} \qquad \text{......(7·43)},$$

where $$m^2c^2 + p_n{}^2 = W_n{}^2/c^2.$$

(*c*) We now rewrite the wave equation (7·12) introducing the values of V and $\mathbf{A}$ from (7·41) and (7·42) but neglecting the terms in a^2. We thus obtain

$$\nabla^2\psi - \partial^2\psi/c^2\partial t^2 - (2\pi mc/h)^2\psi = -(4\pi\iota e/hc) . \cos\phi_1(\mathbf{a}.\operatorname{grad}\psi) \qquad \text{......(7·44)}.$$

In the right-hand side of this equation we substitute, as a first approximation, the value of ψ (7·43) for the undisturbed wave system. Then

$$\nabla^2\psi - \partial^2\psi/c^2\partial t^2 - (2\pi mc/h)^2\psi = (8\pi^2 e/h^2 c)(\mathbf{a}.\mathbf{p}_n)\psi_n\cos\phi_1.$$

The solution of this equation is

$$\psi = \psi_n(1 - 2\iota\delta\sin\phi_1) \qquad \text{......(7·45)},$$

where δ is a constant, depending on $\mathbf{a}$.

(*d*) The complete wave function therefore will consist of the sum or integral of a number of terms of the form

$$\alpha_n\{\exp\iota s_n - \delta\exp\iota(s_n + \phi_1) + \delta\exp\iota(s_n - \phi_1)\},$$

where $$hs_n = 2\pi\{(\mathbf{r}.\mathbf{p}_n) - W_n t\}.$$

When the charge and current density are calculated by means of equations (7·32) and (7·33) the resulting expressions may each be divided into three parts.

(i) The first part contains terms independent of δ and gives the original charge and current due to the free electrons.

(ii) The second part contains terms involving δ linearly, and involves the coordinates $\mathbf{r}$ and t only in terms of the form

$$f_{mn}(t) = \cos(s_m - s_n \pm \phi_1).$$

(iii) The third part contains terms involving δ^2 and will therefore be neglected.

(*e*) The radiation emitted is determined as in equations (6·15) by expressing the potentials for the scattered light as integrals of the form

$$\int f_{mn}(t - R/c) \, . \, d\tau'/R,$$

where
$$R = |r\mathbf{n}_2 - \mathbf{r}'| = r\{1 - 2(\mathbf{r}' . \mathbf{n}_2)r^{-1} + (r'/r)^2\}^{\frac{1}{2}},$$

and $\mathbf{n}_2$ is a unit vector giving the direction of scattering considered. When we consider the scattered radiation at a great distance from the scattering system we may write

$$R \doteqdot r - (\mathbf{r}' . \mathbf{n}_2),$$

but since the scattering system is no longer of atomic dimensions as in § 6·2 we must not make the approximation $R \doteqdot r$. The argument of f_{mn} may now be written as

$$(2\pi/h)\{\mathbf{r}' . \mathbf{p}_m - \mathbf{p}_n \pm \mathbf{f}_1\} - (2\pi/h)\{W_m - W_n \pm E_1\}(t - r/c)$$
$$- (2\pi/h)(W_m - W_n \pm E_1)(\mathbf{r}' . \mathbf{n}_2)/c.$$

The integrals involving f_{mn} evidently represent a radiation of frequency

$$\nu_2 = (E_1 \mp W_m \pm W_n)/h.$$

If we write
$$E_2 = h\nu_2, \quad \mathbf{f}_2 = h\mathbf{n}_2/\lambda_2, \quad \lambda_2 = c/\nu_2 \qquad \text{......(7·46)},$$
the argument of f_{mn} becomes

$$(2\pi/h)\{\mathbf{r}' . \mathbf{p}_m - \mathbf{p}_n \pm \mathbf{f}_1 \mp \mathbf{f}_2\} \pm (2\pi\nu_2)(t - r/c).$$

Hence the mean value of f_{mn} taken over the whole of the scattering system will be zero unless

$$\mathbf{p}_m - \mathbf{p}_n = \pm(\mathbf{f}_2 - \mathbf{f}_1) \qquad \text{......(7·47)}.$$

This equation, together with

$$W_m - W_n = \pm(E_2 - E_1) \qquad \text{......(7·48)},$$

determines the frequency of the light scattered in the direction defined by $\mathbf{n}_2$.

But these are precisely the equations which express Compton's original theory of this phenomenon. Take the upper sign and write $m = 1$, $n = 2$. Then equation (7·47) equates the loss in the momentum of a quantum, $\mathbf{f}_1 - \mathbf{f}_2$, to the gain in the momentum of a scattering electron, $\mathbf{p}_2 - \mathbf{p}_1$. Equation (7·48) equates the loss in the energy of the quantum, $E_1 - E_2$, to the gain in the energy of the scattering electron, $W_2 - W_1$. Hence the wave-length of the scattered light determined by wave mechanics must agree with that determined by Compton's theory.

DISCUSSION

The President: I must congratulate the author on his manner of presenting a very difficult subject and thank him for the new angles of view which he has shown us.

THE EFFECT OF SUPERIMPOSED MAGNETIC FIELDS ON DIELECTRIC LOSSES AND ELECTRIC BREAKDOWN STRENGTH

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ABSTRACT. A description is given of tests made to determine the effect of superimposed magnetic fields on the insulating properties of dielectrics. As a result of the tests made the author reaches the conclusion that both the electric strength of dielectrics and the dielectric losses in the same are seriously affected by the presence of superimposed magnetic fields. A theoretical explanation of the phenomena is suggested by a paper read by Professor A. Smouroff before the Mathematical Conference at Bologna in September 1928. The investigation is not complete and the author is proceeding with dielectric loss measurements using more refined laboratory apparatus than that which he had available for the series of tests described.

During the operation of plant on three extensive 115,000 volt networks involving some 300 miles of transmission lines and heavy generating and transforming plant a number of phenomena were observed in connection with insulation failures and corona discharges which did not yield to ready explanation, and consequently the idea was suggested by the author's colleague, Mr L. C. Thornton, that dielectrics when simultaneously subjected to a magnetic field superimposed on a dielectric field had their insulating characteristics appreciably affected by the presence of the magnetic field.

A search through the available literature on the subject of dielectrics yielded no experimental evidence of such phenomena having been previously observed or discussed in relation to solid dielectrics. The effect of the presence of a magnetic field on the electric strength of various gaseous dielectrics has however been investigated by numerous authorities. From these investigators' experiments the fact seemed to have been established that the presence of a magnetic field tends to increase slightly the sparking or breakdown voltage of gaseous dielectrics under most practical conditions.

In order to make a preliminary test to investigate this phenomenon the apparatus shown in Fig. 1 was set up. This apparatus consisted of a B.E.S.A. Standard insulating oil test cup with one electrode extended so that it constituted the core of an electromagnet. A large number of tests were made on different oils using both A.C. and D.C. magnetic fields. In the case of alternating current the phase relationship of the dielectric field to the magnetic field was not observable but calculation showed that there was approximately no phase difference between these fields. The results obtained are tabulated in Table I.

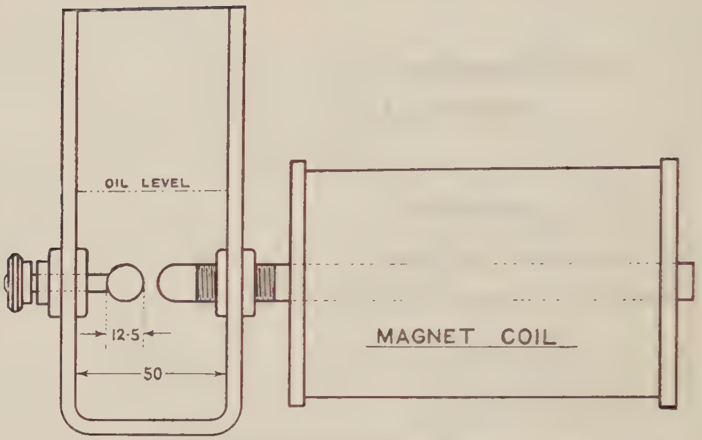

Fig. 1. Apparatus for preliminary investigation.

Table I: Breakdown voltage.

Current: 50 $\sim$ A.C. *at* 15·5° *C.*

	1st Series			2nd Series	
	Without magnetic field	With A.C. magnetic field 1·1 amps excitation	With A.C. magnetic field 2·0 amps excitation	Without magnetic field	With D.C. magnetic field core saturated
No. of tests	10	10	10	10	10
Average breakdown voltage	46222	43111	38888	37875	35500

3rd Series					
	Without magnetic field	With A.C. magnetic field 2·0 amps excitation		Without magnetic field	With A.C. magnetic field 2·0 amps excitation
No. 1 oil sample:			No. 3 oil sample:		
No. of tests	10	10	No. of tests	7	7
Av. breakdown volt.	48450	43450	Av. breakdown volt.	31714	30500
No. 2 oil sample:			No. 4 oil sample:		
No. of tests	5	5	No. of tests	6	6
Av. breakdown volt.	48375	41375	Av. breakdown volt.	40750	34666

From the above table it would appear that the influence of the magnetic field was greater in the case of the drier oils, i.e. those which broke down at a higher voltage.

This result led to experiments on an ordinary commercial dielectric of relatively poor insulating characteristics, i.e. pressboard.

The tests on pressboard were made with standard E.R.A. electrodes with the difference that the lower electrode was made of steel and constituted the pole of an electromagnet. In making the tests the voltage was raised to what was known to be approximately the one minute breakdown value for the material under standard conditions at 15·5° C. in air and the average time before breakdown occurred under various magnetic field conditions was recorded. The results of these tests are shown in Table II.

Table II: Breakdown voltage on 3 mm. pressboard.
(Material of German manufacture).

1ST SERIES

	Without magnetic field	With 50 ∼ A.C. magnetic field 2·0 amps excitation
(*a*)	10000 V—intact 4 min.	10000 V—intact 4 min.
	12000 V—failed 1·5 min.	12000 V—failed 15 sec.
(*b*)	12000 V—intact 2·5 min.	12000 V—intact 2·5 min.,
	14000 V—failed 15 sec.	failed whilst raising volts at 13500 volts
(*c*)	14000 V—failed 100 sec.	14000 V—failed 20 sec.
(*d*)	14000 V—failed 15 sec.	14000 V—failed 6 sec.
(*e*)	14000 V—failed 75 sec.	14000 V—failed 67 sec.

2ND SERIES

	10000 volts	15000 volts	Rising 1000 V per 1 sec.
Without magnetic field	2 intact	2 intact	Failed 18000 V
With D.C. magnetic field (full saturation)	2 intact	1·5 failed	
Without magnetic field	2 intact	2 intact	Failed 19500 V
With D.C. magnetic field (full saturation)	2 intact	0·25 failed	

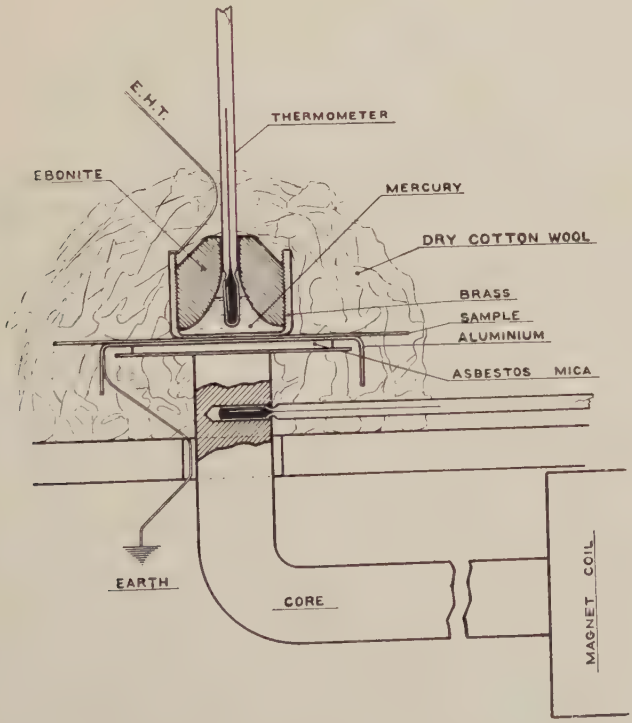

Fig. 2. Apparatus used for results shown in Fig. 3 and Table III.

These results led the author to the view that the magnetic field under certain circumstances had a very marked effect on the dielectric losses in the material and consequently on the ultimate breakdown strength. To check this theory the apparatus shown in Fig. 2 was set up since the conditions and facilities available did not permit of any more accurate method being employed. This test constitutes a modification of the "highest maintained A.C. stress" test described in E.R.A. Report A/S 2*.

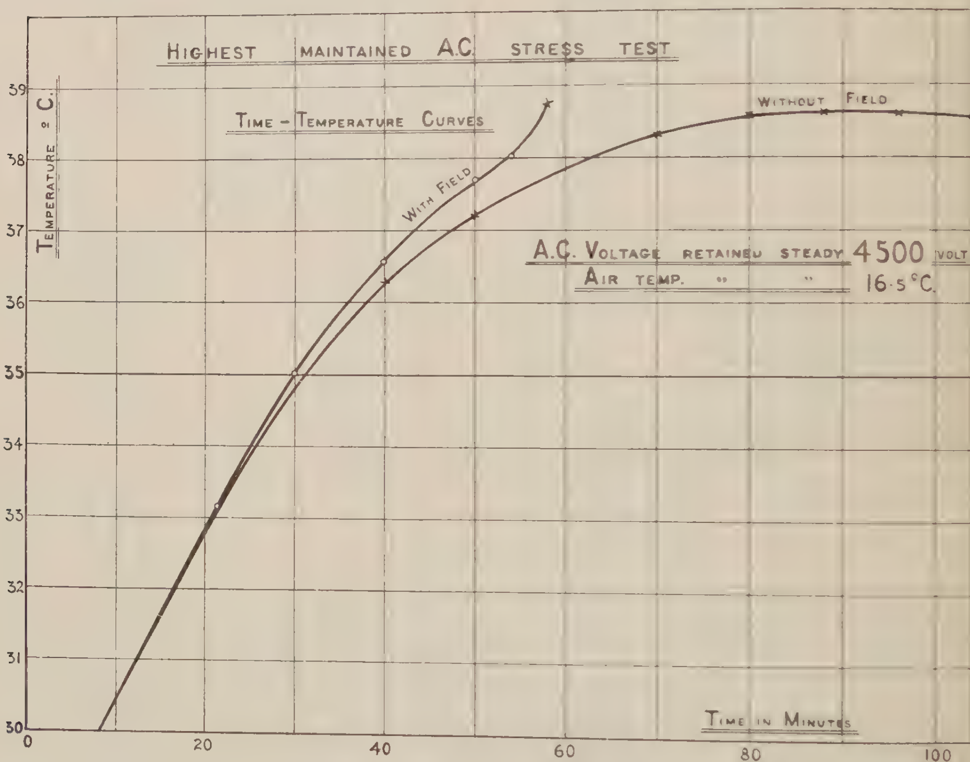

Fig. 3. Tests on varnished cloth.

The material tested was a double thickness of 7 mm. varnished cloth. The curves in Fig. 3 show the average of many tests made. The steep form of the curve, when the magnetic field is superimposed on the dielectric field, suggests a most marked increase of the dielectric loss in the material.

In making this test particular care was taken to isolate the sample from temperature changes due to the heating of the magnet employed. The magnetising coil was arranged at a distance and the temperature of the core was observed and maintained constant.

Following this a number of simple breakdown strength tests (employing standard E.R.A. rules for conditioning and testing) were carried out on samples of varnished cotton cambric. These results are tabulated in Table III.

* *I.E.E. Journal*, **60**, 794–802 (1922).

Table III

Time of breakdown in varnished cotton cambric

1st Series. Single Thickness

Electrodes standard E.R.A. in form but lower one of aluminium. A.C. voltage of 5500 volts (50 ∿) was supplied and time of breakdown observed in seconds.

(a) Without magnetic field	(a) With D.C. magnetic field	(b) Same test repeated Without magnetic field	(b) Same test repeated With D.C. magnetic field	(c) Same test repeated Without magnetic field	(c) Same test repeated With D.C. magnetic field
59	75	65	49	14	6
65	23	52	14	20	15
35	15	13	35	40	16
159	113	35	13	23	42
Av. 53	38	33	28	36	32
		45	25	133	111
		25	30	Av. 26	22
		268	194		
		Av. 38	27		

2nd Series. Double Thickness

Electrodes standard E.R.A. in form but lower one of aluminium. Voltage applied 10,000 volts.

Without magnetic field	With D.C. magnetic field
124	20
40 (bad crinkled sample)	82
421	37
30	52
115	
730	191
421 (omit as a freak point)	Av. 48
309	
Av. 77	

3rd Series. Double Thickness

Upper electrode standard E.R.A. in form but lower electrode a copper bar through which 2000 amps A.C. was passed to produce a field round it. Tests recorded on the same line below were made at adjacent places on the sample.

No. 1 Material (German origin), Voltage applied 9800 V — Time (seconds) to produce failure: Without current in electrode	With 2000 amps passing	No. 2 Material (Swedish origin), Voltage applied 11,300 V — Time (seconds) to produce failure: Without current in electrode	With 2000 amps passing
60	25	350†	60
40	5	20	7
400*	30	40	10
500	60	410	77
Av. 166	20	Av. 137	26

* No failure.

† No failure. This sample was retested with field on and again produced no failure. Examination showed it to be a sample in which the varnish was thicker than normal.

Note. The results in 3rd Series of Table III (although they appear to indicate that the field surrounding a conductor has an effect on the insulation) cannot be regarded as satisfactory. The material employed was not sufficiently homogeneous.

The above described tests seemed to confirm the suggestion that a magnetic field superimposed on a dielectric field increased the dielectric losses and weakened the electric breakdown strength of solid dielectrics and liquid dielectrics (as represented by transformer oil).

A theoretical explanation for the phenomena was suggested by the contents of a paper by Professor A. Smouroff*. Professor Smouroff's explanation of the mechanism of electric breakdown appeared to be capable of extension to explain the observed phenomena.

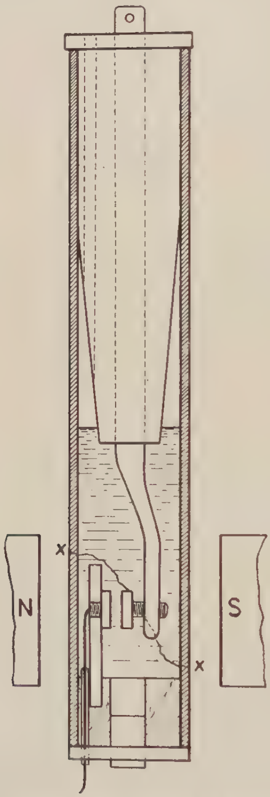

Fig. 4. Abandoned design of apparatus showing fracture $x..x$.

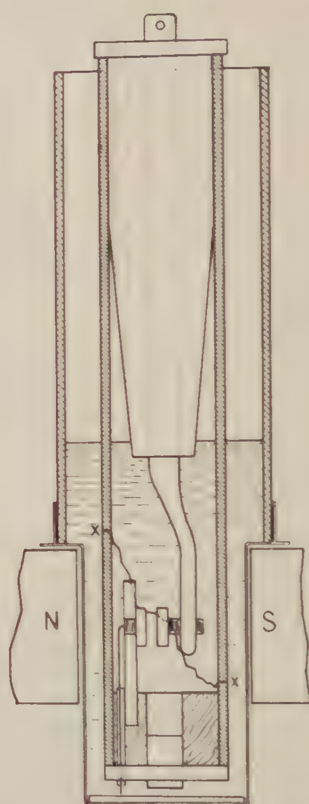

Fig. 5. Abandoned design of apparatus showing fracture $x..x$.

Consequently the author approached Prof. Smouroff and the following practical tests have up to date been carried out conjointly by Prof. Smouroff and the author in the laboratories of the Electrotechnical Institute in Leningrad, using the special apparatus of the author's design shown in Figs. 4, 5 and 6.

The fact seems worth recording that the first form in which this apparatus was designed was as shown in Fig. 4. This apparatus gave perfectly satisfactory results using voltages up to even 50,000 volts as long as no magnetic field was involved.

* "The Physical Nature of Dielectric Breakdown," a paper read before the International High Tension Conference in Paris in 1927.

With a strong magnetic field excited with D.C. current the glass tube fractured at the position marked *xx* when the voltage reached 38,000–40,000 volts. New glasses and specially annealed glass tubes were cut and fitted three times but with the same result each time. The apparatus shown in Fig. 5 was developed with oil surrounding the glass tube in such a manner as to eliminate corona and mechanical pressure, but the same trouble due to the cracking of the glass tube as soon as the voltage reached a value of approximately 40,000 volts with a superimposed D.C. magnetic field was encountered. This set-back in making up the apparatus in itself appears to reveal a phenomenon worthy of further investigation and probably

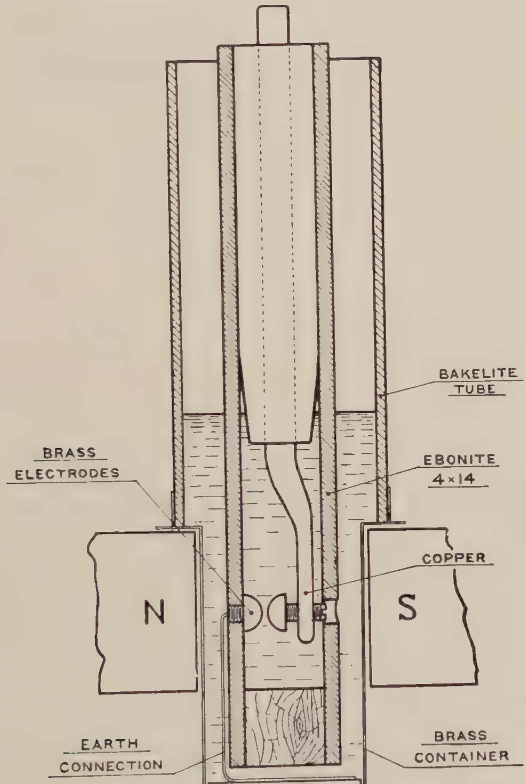

Fig. 6. Improved design of apparatus.

explainable by the theory of the behaviour of electrons in dielectric and magnetic fields which has been advanced by Prof. Smouroff.

The test gap was so made that it could be placed in the air gap of a specially adapted transformer yoke (which was utilised for producing suitable magnetic fields) and could be turned in the gap so that the magnetic field was either transverse or parallel to the dielectric field.

The materials available for preliminary tests were none of them what might be called good dielectrics, hence it was decided to make the preliminary tests described below on:

(1) Air at ordinary laboratory temperature, pressure and humidity.

(2) Transformer oil (commercially dry).

(3) Pressboard (as representing a commercial dielectric and one of comparatively poor quality although extensively used).

(4) Sulphur as representing a good dielectric the properties of which are well known.

TESTS ON AIR

The first results obtained when testing the breakdown strength of air were so extraordinarily contradictory that it was difficult to find any explanation for their divergencies. The results of some 293 tests are represented by the curves in Fig. 7 in which air gap has been plotted against percentage decrease and percentage

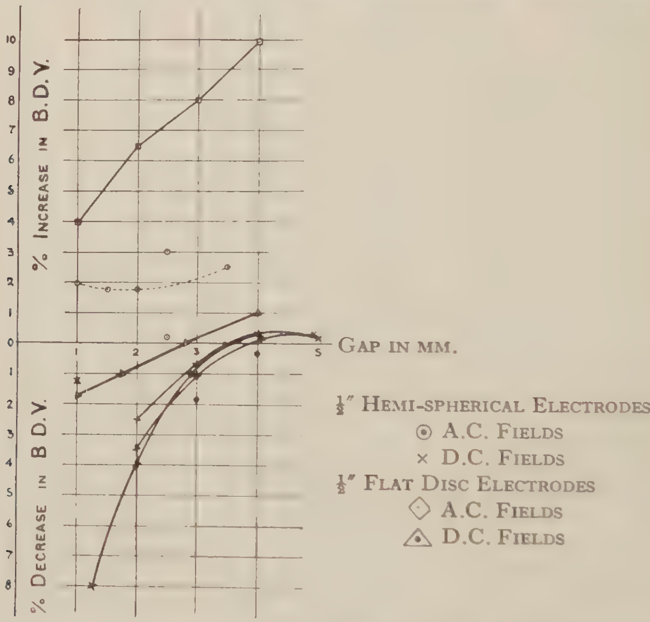

Fig. 7. Effect of transverse magnetic fields on breakdown voltage of air. Laboratory temperature 18·0° C.

increase in breakdown voltage under the influence of various magnetic fields. These curves all show that the length of the spark gap appreciably influences the effect of a superimposed magnetic field on the breakdown voltage of fluids.

This result is of much interest because it would appear that there are other influences at work which tend to mask the real decrease in breakdown strength under the influence of a magnetic field which it is reasonable to expect if the conclusion reached by Prof. Smouroff in a recent paper* be accepted.

The character and the sound of the arc during discharge varied appreciably

* "The Physical Nature of the Phenomena in Homogeneous Dielectrics," read before the International Mathematical Conference at Bologna in September 1928.

when tests were made with and without strong magnetic fields, hence the conclusion was reached that the masking of the decrease in breakdown strength due to the deflection of electrons in their normal orbits (anticipated by Prof. Smouroff) was due to electrodynamic forces deflecting the leakage currents which flow through the dielectrics prior to breakdown and so extending their paths as to increase the sparking voltage. This masking effect might be expected to increase with the length of the gap, as the curves in Fig. 7 show. It is moreover also evident from Fig. 7 that the influence of the spark gap length is greater with A.C. than with D.C. magnetic fields. This too might be expected, as the blow-out effect of an A.C. magnetic field in phase with the dielectric field will be greater than that of a corresponding D.C. field on the same A.C. dielectric field.

Tests were also made with the hemispherical electrodes arranged so that the dielectric field was parallel to the magnetic field. In this case the results (as seen from Table IV) showed a decrease in breakdown voltage for both A.C. and D.C. fields which in the latter case approximately corresponded to that observed with a transverse field.

Table IV: Tests on air.

Gap in mm.	Excitation of magnet	Excitation current in amps	No. of tests	% decrease in electric strength with magnetic field
1·80	A.C.	11	10	2·5
1·95	D.C.	16·4	10	2·5

TESTS ON OIL

The majority of the tests on oil were made with commercially dry oil, i.e. not on oil which had been specially dried to give it an electric strength higher than that obtaining in ordinary electric high tension work.

It was found that on any particular sample of oil the first ten tests made gave results which varied over a wide range, but that after ten tests apparently the moisture in the neighbourhood of the gap had been attracted into the gap and dispersed by the discharges so that the sample improved in strength and finally gave the most extraordinarily uniform results, even although a further 60 or 70 tests might be made. Table V shows a typical example of such a test. In this case

Table V: Tests on oil.

Gap in mm.	Arrangement of gap	Magnetic field current (amps)	Average 5 tests without field	Average 5 tests with field	% increase with magnetic field	% decrease with magnetic field
2	Longitudinal	A.C. 21·5	18500	20800	8·9	—
2	,,	,,	22060	23140	9·5	—
2	,,	,,	23140	23850	9·7	—
2	,,	,,	20720	23400	8·9	—
2	,,	,,	20940	23080	9·1	—
2	Transverse	A.C. 21·5	20500	23060	8·9	—
2	,,	D.C. 16	20400	17550	—	8·6
2	,,	,,	19180	17200	—	9·0
2	,,	,,	20450	18430	—	9·0
2	,,	,,	20400	17500	—	8·6

it will be seen that the A.C. field, whether transverse or longitudinal, had the effect of increasing the breakdown strength of the oil. This increase was however not observed in other tests made with a weaker A.C. field and with greater gap settings. The application of a D.C. field resulted in a decrease of the breakdown value of the oil under test.

The above results of tests on oil confirm the observations made in the tests on air: i.e. the electrodynamic forces acting on the leakage currents through the dielectric before breakdown occurs appear to increase their path and hence increase the sparking voltage so that the decrease in sparking voltage due to the effect described by Prof. Smouroff is masked.

TESTS ON PRESSBOARD

The tests on pressboard were made on specially prepared small samples cut from good quality pressboard and subjected to normal conditioning according to E.R.A. rules before the tests were made.

The results of these tests showed that with both D.C. and A.C. transverse fields there was comparatively little effect produced, but with a D.C. longitudinal field the decrease in the breakdown strength of the material was 27·5 per cent. and in the case of an A.C. field the average decrease in breakdown strength was 23·4 per cent., the figures being as in Table VI.

Table VI: Tests on pressboard.

	Breakdown voltage without field	Breakdown voltage with A.C. field	Breakdown voltage with D.C. field
	23700	18000	15500
	23600	19000	16300
	17400	16800	19400
	18400	17700	16400
	23900	14800	14600
	23600	14000	14400
	20000	18600	16000
	23500	17200	
	24000		
	24000		
	222100	136100	112600
Average	22210	17012	16085
Percentage decrease with magnetic field		23·4	27·5

In making this test it was observed that the comparatively small residual magnetism in the core of the test magnet was sufficient to affect the result, and hence the core had to be demagnetized with A.C. before reliable readings of breakdown strength without field could be obtained.

These tests on air, oil and pressboard were sufficiently convincing to indicate that the presence of a magnetic field had a very marked effect on the characteristics of dielectrics, but that the measurement of breakdown voltage as a means of indicating this effect was unsatisfactory because other effects entered into the test which could not be accurately determined. Consequently further tests using sulphur, etc. were

abandoned and plans were made for measurement of dielectric losses in various dielectrics under the influence of both A.C. and D.C. fields of varying intensity. These tests are proceeding and will form the subject of a subsequent paper. Preliminary tests have shown that under certain conditions the dielectric losses may be increased in a dielectric as much as 30–35 per cent. by the presence of a magnetic field. The mathematical treatment of the subject by Prof. Smouroff in his paper referred to above would indicate that effects of this magnitude are to be expected.

A NEW ALTERNATING CURRENT POTENTIOMETER OF LARSEN TYPE

By ALBERT CAMPBELL, M.A.

Received November 8, 1928. *Read and discussed November* 23, 1928.

ABSTRACT. The instrument is a modification of the simple Larsen A.C. Potentiometer, in which the unknown voltage is balanced by another (v_2) of the form $(r + j\omega m)\,i_1$, where i_1 is a constant standard current and r and m are resistance and mutual inductance, both variable. In the simple type the quadrature component $\omega m i_1$ is obtained by multiplying the observed $m i_1$ by the pulsatance ω (i.e. $2\pi \times$ frequency). The new system avoids this trouble by ensuring that at any given frequency the current through the primary of the mutual inductance shall be such a multiple of i_1 as to make the balancing voltage $(r + jbM)\,i_1$ where M is the mutual inductance and b a constant. This is achieved by an arrangement of loop-shunt, which can be set for any given frequency so that the readings of r and bM shall give the two components directly in millivolts. The standard alternating current i_1 is set and held constant by a thermal null method in which it is balanced against a known direct current.

§ 1. HISTORICAL INTRODUCTION

MANY years ago the author showed* how impure mutual inductances (and iron-cored current transformers) could be tested by balancing the secondary voltage by means of the simple combination of variable resistance r and mutual inductance m shown in Fig. 1.

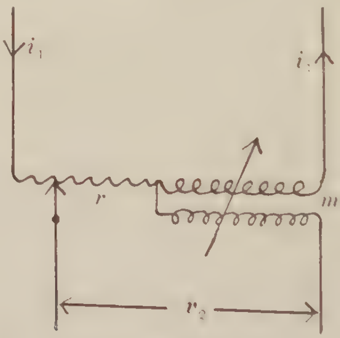

Fig. 1. Simple Larsen potentiometer.

Not long afterwards Sharp and Crawford† published methods of testing current transformers, which utilised a similar combination of r and m, and one of their elegant methods still remains the best for the purpose.

* A. Campbell, *N.P.L. Report for* 1908 (March 1909) and *Proc. Phys. Soc.* 22, 214 (1910).
† C. H. Sharp (In *Discussion*, June 30, 1909), *Am. I.E.E. Trans.* 28 (2), 1040 (1910) and later.

From the author's particular application of the (r, m) combination, Professor Larsen almost immediately recognised its much more general use and turned it into his beautifully simple A.C. potentiometer*, in which an unknown voltage is balanced by v_2 (Fig. 1) which is equal to $(r + j\omega m)\, i_1$ where i_1 is an alternating current held constant in magnitude and phase.

§ 2. THE NEW SHUNT SYSTEM

In the simple Larsen system the in-phase and quadrature components of the unknown voltage are given by ri_1 and $\omega m i_1$ respectively, and so, to obtain the latter, it is necessary to multiply the m-reading by the pulsatance ω, got by a knowledge of the frequency. The main object of the new system is to avoid this multiplication and thus make the instrument read the two component voltages directly.

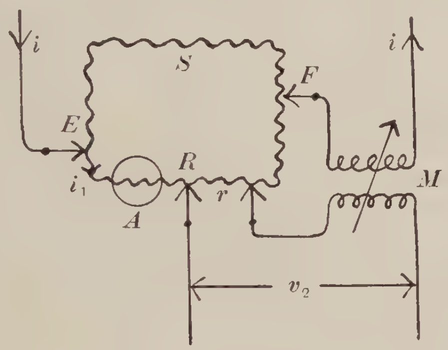

Fig. 2. Loop-shunt system.

The arrangement by which this simplification is attained† is shown in Fig. 2. The supply current (of instantaneous value i) divides itself between the branches R and S of a loop-shunt with sliders keeping $(R + S)$ constant; it then passes on through the primary (invariable) circuit of the mutual inductor, which by adjustment of its secondary circuit can give continuous variation of M from a small negative value up to a large positive one. The in-phase component is taken across r, which is of ordinary potentiometer type with appropriate sliders and change-over switches, forming part of the arm R, which also includes a current-measuring device A, to be described below. Let i_1 be the current in the arm R, and v_2 the resultant voltage for balancing the unknown one. Also let the constant $(R + S) = 2a$.

Then $$v_2 = ri_1 + j\omega Mi \qquad \text{......(1)}.$$

Now $$i = i_1 (R + S)/S;$$

so that if the ratio of S to R is set so as to make $S/(R + S)$ equal to ω/b, we have

$$v_2 = (r + jbM)\, i_1 \qquad \text{......(2)},$$

* A. Larsen, *Elekt. Zeits.* 12, 1039 (1910).

† British Patent Specification 294,053.

where b is constant for this setting. When the frequency is known, it is only necessary to set the loop-sliders E and F so that

$$S = 2a\omega/b \qquad \text{......(3)},$$

and S can be graduated to read directly in n or in ω. It will be noticed that the loop-shunt system is ideal for this purpose, as it only requires the resistance S to be set directly proportional to the frequency. The constant b is chosen so as to make the inductor part read in the same units as the r-part.

§3. STANDARDISATION OF THE REFERENCE CURRENT

In order that the readings of r and M may give absolute values for v_2, the reference current i_1 must be set and held at a constant known value. This is done by comparison with a standard direct current (set by resistance and standard cell), using the device marked A in Fig. 2. It works by an extension (to alternating current) of a null method which the author has already published*, and is shown in Fig. 3. It consists of a heater H, a thermopile Th with d.c. galvanometer G,

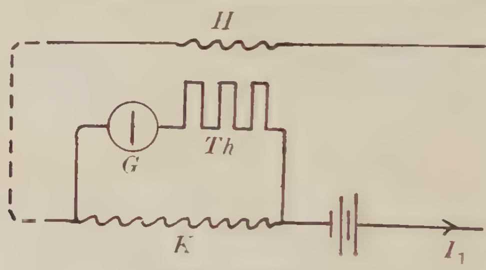

Fig. 3. Current-standardisation circuits.

and a low resistance K. In order to set and hold the alternating current i_1 at a desired value I_1, direct current of the value I_1 is first sent through H and K, K being of such a value as to give zero reading on the galvanometer. (The voltage drop in K balances the thermal voltage in Th.) Then the direct current is switched off from H through an equal resistance so as still to maintain its value I_1 through K, and the alternating current i_1 is sent through H and adjusted until balance is again attained on the galvanometer. When this occurs, i_1 will have the correct value I_1 and, as long as the auxiliary direct current remains unchanged, can be held at this value by keeping the galvanometer deflection zero. An accuracy of 1 part in 1000 can be got if the alternating current source is steady enough. The standard reference current i_1 is 10 milliamperes.

§4. CONSTRUCTION AND CAPABILITIES OF THE INSTRUMENT

The actual instrument, which is made by The Cambridge Instrument Co., is shown in Fig. 4. The r-part reading the in-phase component has a range up to 1·8 volts and can be read to about 0·00001 volt; it can also be used as a direct

* *I.E.E. Journal*, **30**, 889 (1901).

current potentiometer. The *m*-part, for the quadrature component, consists of one of the author's new type inductometers with scale of constant percentage accuracy. It has extremely small impurity and can be used as a separate instrument. It gives a range up to 1 volt, readable to about 10 microvolts; for very small values it can be read to about 2 microvolts. Thus, by the use of a resistance of 100 ohms, currents of a few microamperes can be measured.

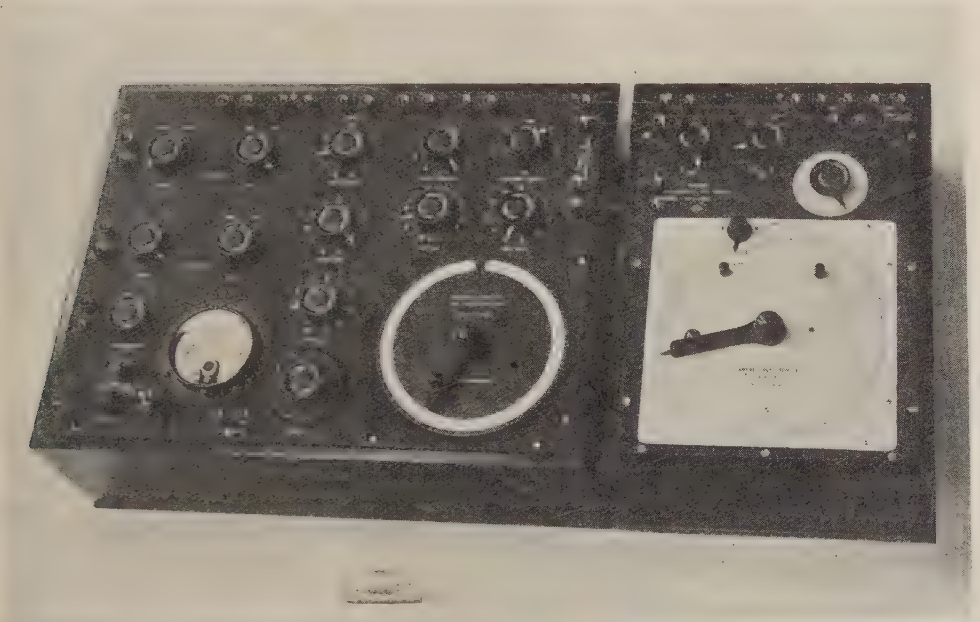

Fig. 4. Campbell-Larsen A.C. potentiometer.

The frequency-setting part can be set with an accuracy of at least 1 in 1000 at any part of its range from 25 up to 1000 cycles per sec. Higher frequencies require a simple multiplier to be used. Coarse and fine rheostats for the alternating current and direct current circuits are included in the instrument, and there is a switch which reverses one or both of the components of v_2, indicating automatically in which quadrant the phase-angle $\tan^{-1}(M/r)$ lies. The detecting instrument is usually a telephone or vibration galvanometer.

The author's best thanks are due to Mr W. L. Beck for the skilful manner in which the details of construction have been planned and carried out.

Note added November 25, 1928

Reference should be made to two other potentiometers in which the Larsen system is used with devices somewhat similar to the author's arrangement: W. Geyger, *Arch. f. Electrot.* **13**, 80 (1924), and A. Pagès, *Société d'Études pour Liaisons Téléphoniques etc.*, Sept. 25, 1925.

DISCUSSION

Dr Drysdale: I must congratulate Mr Campbell heartily on his great improvement on the Larsen potentiometer and I am glad to see that he has brought the subject of A.C. potentiometers again before the Society, as their value has never been sufficiently appreciated in this country. A.C. measurements are now the most important part of electrical testing and involve a very large amount of costly apparatus, so that a single instrument which enables nearly the whole range of such tests to be made within an accuracy of one or two tenths per cent., using only a simple volt box and set of non-inductive "shunts" seems to be most desirable. This has been appreciated in many countries, more especially in Japan, but there still seems to be an impression in this country that A.C. potentiometers are complicated and difficult to use, and of doubtful accuracy. Modern A.C. supplies however are so good both for wave-form and constancy of frequency that the principal objections to the use of A.C. potentiometers have largely disappeared, and it is to be hoped that their extreme adaptability will become more generally appreciated.

The Larsen form of potentiometer is certainly the most simple, and Mr Campbell's modification removes the principal objection to it (that of variation of the quadrature component with frequency), and makes it direct reading. Mr Gall's "co-ordinate" potentiometer secures the same result by preliminary adjustment of the phase-splitting device; but the Pedersen potentiometer, in which the quadrature relation is independent of frequency, cannot easily be made direct reading for different frequencies.

All these instruments are however on what may be called the rectangular or Cartesian principle in which the components x and y of the vector voltage are measured and its magnitude is obtained as $\sqrt{x^2+y^2}$ while the phase angle θ is obtained as $\tan^{-1} y/x$. In actual testing it is much more convenient to measure the magnitude and phase of the voltage directly, and the Drysdale-Tinsley potentiometer which effects this would still seem to be the most useful all-round instrument, although there are a few cases, such as the measurement of very small phase angles nearly in phase or in quadrature with the supply, where the rectangular principle is more accurate. But the comparison of the two methods can best be visualised by realising that A.C. measurements are analogous to surveying, and that the rectangular co-ordinate principle is equivalent to being able only to make measurements, say of a house, along N.S. or E.W. directions, instead of being able to measure off the length along the walls directly and finding their directions (if necessary) by a theodolite. The phase-shifting transformer of the polar potentiometer corresponds to the theodolite; and as it is provided with cross pointers which enable the cosine and sine of the phase angle, as well as the angle itself, to be read off directly, it makes the calculation of the rectangular components or the equivalent resistance and reactance very easy.

I especially admire Mr Campbell's beautiful and effective thermal method of securing constancy of current as it is quite non-inductive and absorbs much less

power than the dynamometer which has been employed in my own potentiometer. On the other hand, the dynamometer does not require an auxiliary constant direct current.

I hope that Mr Campbell's interesting paper will prove a stimulus towards the employment of A.C. potentiometers in all testing laboratories.

Dr A. H. Davis: Such a potentiometer should be very suitable for the type of acoustical work in which the distribution, transmission, reflection, etc. of sound is measured by means of electrical microphones, the source of sound being a loud-speaker actuated by A.C. of pure wave-form. Mr Campbell's potentiometer measures E.M.F.'s of the magnitude and frequency ranges ordinarily met with in work of the kind, and the supply current i for the potentiometer can be obtained direct from the loud-speaker circuit. The potentiometer null method of measurement is especially advantageous where irregular extraneous disturbances are picked up by the leads from neighbouring X-ray outfits, power mains, etc. Being transients they cannot be satisfactorily eliminated by tuning devices, but with a telephone as detector it is possible to listen to the disappearance of the test note even in the presence of interference which would otherwise be prohibitive. Mr Campbell's loop shunt neatly facilitates the calculation of the phase and resultant from the readings of the instrument, and the operation can be performed readily—without summing squares—on an ordinary slide rule having trigonometric functions on the slider. For θ is found as $\tan^{-1} Y/X$ and then R is found as $Y/\sin\theta$.

The President: The subject of A.C. measurement has been so important for many years that a vast amount of study has been given to it, and one would think that nothing more could be done by the mere devising of circuits; but Mr Campbell has shown that the subject is not exhausted, that improvements can still be imported into apparatus and methods, and, what is noteworthy, his new improvement makes for simplicity and not for complexity.

Author's reply: Dr Drysdale has very clearly indicated the differences between the two types of A.C. potentiometer. Each type has advantages for particular kinds of test work. The Larsen type is of special advantage at the higher audio-frequencies, as it requires no phase-splitting device and uses a very small amount of power for its operation. I agree with Dr Davis that it is ideal for many acoustical investigations. I would emphasise the importance of the instrument for the accurate measurement of very small voltages and current (e.g. a few micro-amperes). In both cases the scale does not follow the square law, but gives direct proportionality, which is an enormous gain at the low readings.

FERROMAGNETIC FERRIC OXIDE

By Prof. E. F. HERROUN, F.I.C.,

and

Prof. E. WILSON, Wh.Sch., M.Inst.C.E., M.I.E.E.

Received September 25, 1928. *Read and discussed November* 23, 1928.

ABSTRACT. A brief account is given of work relating to the above subject that has been published during the past eight years. The authors confirm the observation by Messrs. Sosman and Posnjak that lepidocrocite, but not göthite, yields on dehydration a strongly ferromagnetic ferric oxide. As all the specimens of lepidocrocite examined contained three or four per cent. of manganese oxide, they suggest that this substance may be an essential constituent of this crystalline form of the hydrate. They find that the temperature at which the ferromagnetic oxide is permanently transformed into the common paramagnetic kind is largely dependent upon its mode of preparation.

A table showing the magnetic force at which the magnetic permeability and susceptibility reach their maximum values as well as the coercive force and remanent magnetisation is given for different kinds of magnetic ferric oxide and for magnesium and copper ferrites in the form of compressed rectangular bars. Comparative figures are given for two varieties of natural magnetite. Curves are also given showing the variation of permeability with field strengths ranging from 25 to 1300 c.g.s. units.

Although copper ferrite has a higher maximum permeability than ordinary precipitated magnetic oxide of iron, it falls far below that of the purer forms of native magnetite. Analyses of several natural magnetites are given, and the high or low values of the susceptibility of the ferric oxide resulting from their oxidation are attributed to the presence or absence of impurities, particularly magnesia which forms a magnetic ferrite.

§1. INTRODUCTION

In a paper read to this Society in 1921 on "The Magnetic Susceptibility of Certain Natural and Artificial Oxides*," we drew attention to the very great range of magnetic susceptibility exhibited by ferric oxide in its different forms, and gave references to the discovery of a magnetic form of ferric oxide by Robbins in 1859†, and to the subsequent observations of Malaguti‡ and Liversidge §.

During the past few years further work on this subject has been published; in 1925 L. A. Welo and O. Baudisch‖, using a ballistic galvanometer method in which the oxides in the form of powder were packed "by jarring" in a glass tube 20 cms. long, compared the permeability of precipitated magnetic oxide, Fe_3O_4, with magnetic Fe_2O_3 derived from it by heating the former to 220° C. in air. They found

* *Proc. Phys. Soc.* **33**, 196–205 (1921).
† *Chemical News*, **1**, 11, 1859.
‡ *Ann. de Chim. et de Phys.* ser. 3, **69**, 214.
§ *Trans. Aust. Assoc. Hobart Meeting*, 1892.
‖ *Phil. Mag.* **50**, 399–408 (1925).

an increase in the maximum permeability from 2·93 to 3·39 after this conversion and noted that heating in nitrogen to 550° C. caused a reduction in permeability to near the value for ordinary normal haematite, while magnetite could be heated in an inert atmosphere to 800° C. without loss of permeability after it regained normal temperature. They conclude that the extra oxygen required to convert $2Fe_3O_4$ into $3Fe_2O_3$ can be introduced without undue strain, leaving the structure essentially unaltered. Figures derived from X-ray analysis are given in support of this view of the identity of crystal structure of Fe_3O_4 and ferromagnetic Fe_2O_3.

Later in the same year (1925) R. B. Sosman and E. Posnjak* published, under the same title as this communication, the results of their investigation of a naturally occurring ferromagnetic ferric oxide, found in Shasta County, California, which they received from Graton and Butler. The fact that they regarded this discovery of a natural form of magnetic ferric oxide as very unusual and noteworthy increases our reluctance to accept the statement of S. Hilpert† that "one finds in nature large deposits of strongly magnetic Fe_2O_3 derived from the oxidation of ferrous compounds." It is a matter for regret that Hilpert did not specify any locality where this native magnetic ferric oxide was known to occur. Our own experience and examination of iron ores extending over many years inclines us to the belief that Sosman and Posnjak's view that it is very exceptional is the more correct one. These authors have also found that on dehydrating by moderate heating the two crystalline forms of native monohydrated ferric oxide, göthite and lepidocrocite, the former yielded only the paramagnetic form, while the latter gave a strongly ferromagnetic ferric oxide. This difference in behaviour they suggest is due to difference in the initial crystalline form. Without necessarily accepting this or any other theoretical explanation we can confirm the fact of the marked difference in behaviour of these two crystalline hydrates. These authors have also compared the magnetic susceptibility of both the native ferric oxide and that produced artificially by oxidising precipitated magnetic oxide, with finely powdered natural magnetite from Mineville, N.Y. They do not appear to have found any marked difference in susceptibility between natural and artificial magnetite, nor between them and the magnetic form of ferric oxide, but they show that these have different hysteresis loops and thus indicate that the remanence and coercive force of magnetic ferric oxide are higher than those of Mineville magnetite. But it must be clearly realised that had they used some other form of natural magnetite, such as some specimens from Arkansas‡, the results of this comparison would have been quite different—in fact, the reverse of what they say. They emphasise the difficulties of comparing the susceptibilities of powders so soon as a moderately high value has been reached, and we have ourselves suggested the figure for the mass susceptibility of 0·01 C.G.S. unit as about the limit for which any method employing the pull of a non-homogeneous field can be trusted. The factors which they mention, viz. "fineness of grain, shape of grains, size and shape of the charge as a whole and previous magnetic

* *Journn. Wash. Acad.* **15**, no. 14 (Aug. 1925).
† *Deut. Chem. Ges. Ber.* **42**, B 2, 2248–61 (1909).
‡ *Proc. Phys. Soc.* **31**, 311 (1919).

history" are all of importance as there is an "end-effect" of the mass as a whole apart from that of its constituent grains and the distortion of the field consequent upon the introduction of any appreciable mass of highly permeable material. These considerations have led us to abandon the modified Curie balance when comparing the higher susceptibilities dealt with later.

While we are in general agreement with Sosman and Posnjak's paper, there are two statements in it that appear to require qualification. One is that ferromagnetic ferric oxide is completely changed into the paramagnetic form by heating *for a few minutes* to 650° C.; the other that the ferromagnetic form has a distinctive colour "further toward the yellow than the paramagnetic and of a slightly darker shade, and so tends toward a chocolate brown color." Both of these statements may be, and probably are, true when applied to the ferric oxide obtained by oxidising precipitated magnetic oxide in suspension by ammonium persulphate solution, or by prolonged exposure to air in a moist state with or without moderate heating. But examples will be mentioned and exhibited showing that the colour may vary from a full red to the colder yellowish brown and also that some forms of magnetic Fe_2O_3 require a higher temperature than 650° C., or a much more prolonged heating to convert them into the paramagnetic form.

In a paper in the *Comptes Rendus**, R. Chevallier describes his method of preparing ferromagnetic ferric oxide and to some extent his method of measuring or comparing the susceptibilities. He states that the field strength employed was 174 gausses and that he used a ballistic galvanometer method and measured the throw on making or breaking the primary current, but nowhere in the paper is any mention made of the inductive throw produced when no specimen is present (air core reading), nor are his galvanometer deflections given as differences or excesses over the air core throw. We therefore find it extremely difficult to interpret the results of his measurements. He concludes that magnetic oxide and ferromagnetic ferric oxide have substantially the same susceptibility, but that the latter has a greater permanent magnetisation in the ratio of 6·15 : 3·8. He states that this ferric oxide loses its magnetic properties rapidly between 600° and 700° C., but that it never loses them entirely. In fact, if one heats it for a quarter of an hour to near 900° C. one obtains afresh a black, very ferromagnetic powder. This he attributes to the reconversion of Fe_2O_3 into Fe_3O_4 (La décomposition de Fe_2O_3 en magnétite doit alors intervenir et limiter la disparition du magnétisme). This supposed conversion of ferric oxide into magnetic oxide by merely heating *in air* to 900° C. is entirely opposed to all chemical knowledge of the stability of Fe_2O_3 in regard to oxygen, but may have been caused by the accidental introduction of reducing gases which the use of an electric furnace would have avoided. It is of course well known that heating most forms of ferric oxide for an appreciable time to 900° or 1000° C. will permanently change their colour to black or nearly black, but this physical change is not associated with any chemical alteration of the Fe_2O_3 nor in the case of pure oxide with any appreciable increase in susceptibility. The lowest susceptibility for substantially pure ferric oxide we have yet found was $15{\cdot}2 \times 10^{-6}$ in

* *Comp. Rend.* T. **180**, 1473–5 (1925).

the case of a good specimen of colcothar, or rouge, after heating to 1000° C. for half an hour, and its colour was then quite black. The method of preparation of this magnetic form of Fe_2O_3 employed by Chevallier consisted in heating the magnetic oxide—evidently the precipitated form—to about 350° C. when the oxide takes fire in air and this spreads rapidly throughout the mass.

In contrast with this method, H. Abraham and Planiol in a slightly earlier paper in the *Comptes Rendus**, describe the preparation of ferromagnetic Fe_2O_3 by the reduction of Fe_2O_3 to Fe_3O_4 in hydrogen or carbon monoxide at 500° C. This Fe_3O_4 is not pyrophoric but is very easily oxidised and can give either the red ferric oxide ("colcotar") or magnetic ferric oxide; if burnt in air it burns like tinder and gives only the former, but if it be oxidised slowly in a current of air at 200°–250° C. there is no incandescence and the product is a brown powder somewhat yellowish, strongly ferromagnetic, and nearly like magnetite in its magnetic properties. Alteration to non-magnetic Fe_2O_3 does not appear at 600° C. but appears slowly at 650°, and towards 700° the transformation is complete.

In a paper on "The Thermomagnetic study of certain Iron Minerals" by J. Huggett and G. Chaudron†, curves are given showing the variation of susceptibility with temperatures up to 700° C., but the magnitude of the applied magnetic force is not stated. The curve for magnetite shows a rapid fall near 570° C. (this is higher than that generally accepted as the temperature at which magnetite loses its ferromagnetic properties). They state in the paper that göthite heated without previous dehydration shows an increase near 350° C. *at the moment when the water leaves it*, and in this respect it resembles lepidocrocite; but the curve given for lepidocrocite is totally different from any other, showing that a little before 350° and up to just above 570° C. it acquires a very high susceptibility indeed and cannot be plotted on their scale. The statement that göthite after dehydration by heating to 600° C. in a vacuum exhibits a change in the curve at 570° C. thus showing that it contains 0·001 of magnetite is open to criticism. Unless supported by chemical examination this appears to us a mere assumption, and, even if ferrous iron were found after heating in a vacuum to 600° C., it does not follow that it was present in the original mineral, as it might have been formed by the reducing action of traces of organic matter that would be much more effective in a vacuum than on heating in air. No specimen of crystalline göthite examined by us contained any chemically detectable ferrous iron, and the measured susceptibility is so low as to exclude the possibility of 0·1 per cent. of magnetite in the natural mineral.

§ 2. OXIDATION OF MAGNETITE

Precipitated magnetic oxide of iron can be oxidised to Fe_2O_3 while suspended in an alkaline solution by sodium hypochlorite or by ammonium persulphate, or by prolonged exposure in thin layers to the air in a moist state, or by moderate heating in a current of air. If previously dehydrated by heating in CO_2 or nitrogen to a temperature of 400° to 500° C. after cooling it is more stable, but oxidation can

* *Comp. Rend.* T. **180**, 1328–9 (1925).
† *Ibid.* **186**, 694–6 (1928).

still be brought about by heating in air to about 350° to 400° C. Native magnetite is much more resistant to oxidation, even when reduced to very fine powder in an agate mortar, and different varieties exhibit great differences in their resistance to oxidation*. Some forms require to be heated for some hours to a temperature of 1000° C. in order to oxidise them completely.

The difference in the magnetic susceptibility of the resulting ferric oxide, even when the latter was shown by chemical examination to be free from ferrous iron, presented a very puzzling problem. A purchased specimen of precipitated Fe_3O_4 was found on oxidation to give a highly magnetic ferric oxide as anticipated, but this Fe_2O_3 after heating for an hour at 1060° C. in the open quartz tube of an electric furnace still gave a k_m† value of 3500×10^{-6}, and was "evidently magnetic," i.e., its particles would cling readily to the pole of a small bar magnet. Chemical examination of the Fe_3O_4 from which it was derived showed small but very evident traces of calcium carbonate, indicating that the iron salts from which it was derived had been precipitated by slaked lime. After removal of the calcium carbonate by treatment with dilute acetic or very dilute nitric acid, washing and drying this purified Fe_3O_4, it still yielded an equally magnetic ferric oxide when oxidised in air at about 400° C., but its k_m value now fell to 196×10^{-6} after heating for two hours at 750° C., and after a similar time at 1050° it had fallen to $44{\cdot}3 \times 10^{-6}$. It thus appeared that calcium carbonate, by forming calcium ferrite, might prevent the loss of magnetic properties of ferromagnetic Fe_2O_3 on prolonged heating. Forestier and Chaudron‡ have stated that calcium ferrite is rendered irreversibly non-magnetic by heating to above 700° C. This statement led us to a re-examination of calcium ferrite: some calcium ferrite which had a k_m value of $12{,}230 \times 10^{-6}$ units was heated for thirty minutes at 900° C., allowed to cool and its k_m value again found; it was now 4650×10^{-6} and was "evidently magnetic." It was then subjected for ten hours to a temperature ranging from 1000° to 1050° C., and after cooling its k_m value had fallen to 1120×10^{-6}. It is thus seen that the loss of magnetic properties of calcium ferrite is slow and incomplete, even at 1000° C. Small quantities of calcium salts may therefore give rise to enhanced susceptibility which is difficult to destroy by heating for considerable periods.

§3. NATURAL MAGNETITES

A piece of crystal from Traversella, Piedmont, was reduced to very fine powder in an agate mortar and heated in the electric furnace for four hours at 1000°–1050° C., after which it was examined chemically and showed no ferrous iron; yet its k_m value was over 4000×10^{-6} and it was "evidently magnetic." It was again heated to about 1050° C. for ten hours and after cooling it gave 5410×10^{-6} for its k_m value. The rate of cooling appeared to be influencing the result, and it was found that after heating to 1040° C. for a few minutes and quickly cooling, it gave 6060×10^{-6}, but if cooled very slowly its k_m was 4546×10^{-6}; evidently it was still magnetic, yet completely oxidised.

* Moissan, *Ann. de Chimie* (5), **21**, 233.
† Throughout this paper k_m denotes the mass-susceptibility per gram.
‡ *Comp. Rend.* T. **181**, 509–11 (1925).

Magnetite from Mineville, New York State, U.S.A. Through the kindness of Mr A. M. Cummings, the General Superintendent of Messrs Witherbee, Sherman and Co., we were supplied with some specimens of this magnetite. When finely powdered it oxidised easily in air at 1030°–1060° C. and after three hours' heating at this temperature it was found to contain no detectable ferrous iron. Its k_m value was reduced to 52×10^{-6}, and a further heating at the above temperature merely reduced this already low figure to 47×10^{-6}. This result was in marked contrast with the oxidation product of Traversella magnetite.

Altenfjord (Norway) magnetite similarly treated oxidised fairly easily and gave a product with a k_m value of $71 \cdot 5 \times 10^{-6}$. It contained inclusions of felspar.

Two varieties of magnetite from the Tilly Foster mine, New York, are described elsewhere as Tilly Foster I and II. No. I gave after oxidising at 1000° C. a product with a k_m value of 71×10^{-6}. (This product was dark purplish slate, not quite black.) No. II after four hours' heating at 1000°–1066° C. gave a product chemically free from ferrous iron, but with a k_m value of 8700×10^{-6}. Repowdering and reheating scarcely altered this value, which then became 8110×10^{-6}.

Penryn (Cornwall) magnetite, after heating for four hours as in previous cases, was found to be fully oxidised and its k_m value was 220×10^{-6}.

Arkansas magnetite, after treatment as in other cases, gave an oxidised product with a k_m value of 2760×10^{-6}.

The problem thus presented is why some magnetites like that from Mineville on being fully oxidised gave a ferric oxide of quite low susceptibility (47×10^{-6}), while others such as Traversella and Tilly Foster II gave k_m values of 5000 to 6000 and 8700×10^{-6} respectively. The investigation of this problem involved the complete quantitative analysis of the above named magnetites. All details of analysis are omitted for lack of space, but the general (in some cases, average) results are given in Table I. The small and variable amounts of manganese found in all the specimens did not appear to have any relation to the magnetic properties of the final oxidation product, but a connexion was at once seen when the magnesium content was considered. In Mineville and Tilly Foster I it was absent, while in Tilly Foster II, Traversella and Arkansas it reached the values of 3·0, 1·9 and 4·5 per cent., respectively. That the percentage of MgO in the Arkansas specimens is not in proportion to the retained magnetic properties of the Fe_2O_3 is probably due to the considerable amounts of titanic acid and combined silicates, so that less of the MgO is available for the formation of a ferrite. Probably the presence of the TiO_2 and other impurities may be connected with the very high coercive force (150) and relatively low permeability of this magnetite*. It will be noticed that the ratio (FeO : Fe_2O_3) is higher than that required for pure Fe_3O_4 in this magnetite, thus indicating that a part of the FeO is combined with titanic acid.

In Mineville magnetite the ratio FeO : Fe_2O_3 is correct, showing that apart from unavoidable inclusions of quartz, it is practically pure Fe_3O_4. In the Traversella specimens this ratio is slightly too low, indicating a partial oxidation of this magnetite. The latter condition is very noticeable in Tilly Foster II, where the

* *Phys. Soc. Proc., loc. cit.* (1919).

FeO : Fe_2O_3 ratio falls to 0·368 instead of 0·45, while in Tilly Foster I the ratio is 0·44 or nearly normal. Penryn magnetite is also a partially oxidised one, giving the ratio of 0·375.

Table I: Chemical analysis of natural magnetites

Locality	Arkansas	Mineville, N.Y.	Traversella	Penryn (Cornwall)	Tilly Foster II (N.Y.)
FeO	26·48	30·7	28·4	25·1	25·8
Fe_2O_3	52·00	68·3	66·1	66·9	70·2
SiO_2	2·25*	0·5†	2·34*	7·1*	Traces
TiO_2	9·10	—	—	—	—
MgO	4·5	—	1·9	—	3·0‡
MnO	1·49	0·05	0·103	0·015	0·09
Al_2O_3	4·1	—	—	—	—
	99·92	99·55	98·843	99·115	99·09

* Including insoluble silicates. † As quartz.
‡ Tilly Foster II contained traces of CaO in addition to MgO, and some CO_2 in combination with these bases.

Table I gives the approximate composition of some of the magnetites examined; in each case portions for analysis were obtained as free from visible inclusions as possible. The manganese was in most cases determined by a colorimetric method.

§ 4. GÖTHITE AND LEPIDOCROCITE

These two minerals, which are supposed to have identical composition, namely $Fe_2O_3 . H_2O$, but different crystalline forms have been shown by Sosman and Posnjak, as above stated, to behave quite differently when dehydrated by moderate heating. Powdered göthite from Cornwall was found to give a k_m value of 28·5 × 10^{-6} in its original state. After heating to nearly 500° C. for fifteen minutes, its k_m value was raised to only 33·2 × 10^{-6}, and after half an hour at 750° C., this value fell to 21·1. Two and a half hours' heating at 1050° C. raised it again to the identical figure which it gave originally, namely 32·2; it was darker in colour, but still dark brown and not black.

Another specimen gave an initial k_m value of 30·3 × 10^{-6}; after thirty minutes at 345° C. its k_m value reached 39·8 × 10^{-6}: half an hour at 400° C. reduced it to 34·7 × 10^{-6}, and half an hour at 1000° reduced it still further to 23·8 × 10^{-6}. Thus both these typical specimens behaved similarly, and not unlike non-crystalline brown haematite which had an initial susceptibility of 31·6 × 10^{-6}, falling to 27·3 × 10^{-6} after thirty minutes at 375° C., and falling again to 17·5 × 10^{-6} after fifteen minutes at 750° C., and finally rising slightly to 38 × 10^{-6} after heating for thirty minutes at 1050° C. Its colour was then a dark reddish brown and, like göthite, it did not become black.

Lepidocrocite. The first specimen we examined was from Siegen, Westphalia. Its k_m value in the form of powder before heating was 42 × 10^{-6}; after half an hour at 375° C. it gave a figure of 39,500 × 10^{-6}, i.e. so high that the modified Curie balance, which is excellent for low susceptibilities, became unsuitable and unreliable. After thirty minutes at 750° C. the susceptibility had fallen to 129 × 10^{-6},

but thirty minutes at 1000° C. raised it to 392×10^{-6}: its colour was now a very dark purplish brown, nearly black.

After some delay and difficulty, two other specimens of lepidocrocite from other localities were obtained; one came from Easton, Pennsylvania (the same locality from which the mineral used by Sosman and Posnjak was obtained), the other from Reading, Penn., U.S.A. Their magnetic changes were entirely similar to those of Siegen lepidocrocite, but the k_m value for the Reading specimen was rather lower.

Chemical examination of the specimens of göthite and lepidocrocite disclosed the fact that while in the case of the former there exists only a trace of manganese oxide (0·06 per cent.), in all three specimens of lepidocrocite from these widely different localities there is a considerable percentage of manganese in the form of a higher oxide (i.e. either Mn_2O_3 or MnO_2). In the Siegen specimen it was found to amount to 3·6 per cent., estimated gravimetrically as Mn_3O_4. In both the Easton and Reading specimens manganese is present to roughly a similar extent.

The presence of this amount of manganese accounts satisfactorily for two differences which lepidocrocite exhibits when compared with göthite. (1) The darker colour which its powder acquires after heating. (2) The higher final value for k_m after heating to above 1000° C., namely 392×10^{-6} as compared with 32·2 or $23{\cdot}8 \times 10^{-6}$. Can it account for the high k_m value reached on heating to 350°–450° C.? It does not seem possible that it can do so, not only on account of the very high value that is reached, but also because a mixture of the precipitated hydrated oxides when dehydrated at 400° C. does not give any appreciable rise in k_m, but does so on heating to 750° C., and still more at 1000° C. Thus $MnO.Fe_2O_3$ is not formed at 400°, and when formed, the high k_m value persists even when the specimen is heated for four hours to temperatures above 1000° C. This is entirely at variance with the general behaviour of lepidocrocite, excepting only the retention of a small increase in the final value after 1000° C. over that of pure Fe_2O_3, due, no doubt, to the three to four per cent. of manganese oxide it contains. The suggestion of Sosman and Posnjak that the difference is primarily due to the difference in crystalline form would therefore appear the most probable explanation, were it not for the fact that while precipitated Fe_3O_4 when fully hydrated is oxidised in aqueous suspension, it is practically as magnetic in that hydrated condition as when dehydrated.

Why, then, is not lepidocrocite magnetic as it occurs in nature, and why should it be necessary to dehydrate it? It may be that in its natural state it is not identical in crystal structure with magnetite, but can pass into that form on moderate heating, and it would be an interesting point for mineralogists to determine if three or four per cent. of manganese is essential to the formation of that crystalline variety that is called "lepidocrocite," or if the latter is ever known to occur in pure monohydrated ferric oxide.

Lepidocrocite that has been rendered ferromagnetic by heating for a few minutes at 350° to 400° C. is rapidly converted into the paramagnetic form when the temperature is raised. Heating for 35 minutes to 480° C. gave a reduced k_m value of only 3000×10^{-6}, or less than a tenth of its maximum. It is more easily converted

into the non-magnetic type than artificially produced Fe_2O_3 from Fe_3O_4 by low temperature oxidation, and the latter is less stable than that produced by the oxidation of ferrous oxide obtained by the calcination of ferrous oxalate in a closed crucible with subsequent admission of air, a little at a time. This last is highly magnetic, has usually a fine deep red colour, and is so resistant to heat that in an experiment in which both this oxide and Easton lepidocrocite were treated simultaneously in the same electric furnace, after ten minutes at 650° C. the k_m values were reduced from $22{,}800 \times 10^{-6}$ to 1900×10^{-6} in the case of lepidocrocite, and from $34{,}700 \times 10^{-6}$ to $33{,}500 \times 10^{-6}$ with the Fe_2O_3 from ferrous oxalate. Further heating for ten minutes at 750° C. caused the former to fall to 68×10^{-6} and the latter to 7200×10^{-6}; the latter was now over a hundred times more magnetic than the former. There are thus great differences in the ease with which different forms of ferromagnetic ferric oxide pass into the common non-magnetic form.

Artificial Hydrated Ferric Oxide. Pure precipitated Fe_2O_3 when dried at temperatures below 100° C. may have the composition of $Fe_2O_3 . H_2O$ or, like brown haematite, may be more nearly represented by the formula $2Fe_2O_3 . 3H_2O$. The susceptibilities of these hydrates are always higher than that of the Fe_2O_3 resulting from their dehydration (the exact opposite of the effect in lepidocrocite). The following values have been observed during this investigation: for $Fe_2O_3 . H_2O$, values ranging from 127×10^{-6} to 147×10^{-6}, and for approximately $2Fe_2O_3 . 3H_2O$, 102×10^{-6} to 115×10^{-6} have been measured; while for anhydrous Fe_2O_3, the variation has been from $21{\cdot}1 \times 10^{-6}$ to $33{\cdot}8 \times 10^{-6}$. As these may be regarded as equally pure, the only cause of variation is the mode of preparation having some obscure influence. It seems probable that precipitation at a high temperature with subsequent boiling gives rise to rather higher susceptibility than precipitation at a lower temperature and washing with cold water, but the observations are not conclusive. The minimum value for Fe_2O_3 recorded above is $21{\cdot}1 \times 10^{-6}$, and is slightly higher than that obtained from brown haematite after heating to 750° C. of $17{\cdot}5 \times 10^{-6}$, and as already mentioned, the figure $15{\cdot}2 \times 10^{-6}$ for a black form resulting from heating a very good sample of rouge to 1000° C. is the lowest we have ever recorded for Fe_2O_3 in any form. It is curious that the susceptibility of normal Fe_2O_3 should be less than that of any of the oxides of Mn, Co or Ni.

Purchased hydrated ferric oxide is unfortunately in our experience unsuitable for magnetic work; besides containing frequently an appreciable quantity of calcium carbonate, it sometimes contains a trace of some organic substance (possibly derived from methylated spirit) which when the oxide is heated in a confined space, or in a large bulk, reduces a small portion of the ferric oxide to magnetite and so greatly enhances the susceptibility. Heated in a thin layer on a crucible lid it does not show this effect, or scarcely so. When it is remembered that one part of carbon can reduce eighty times its own weight of Fe_2O_3 to Fe_3O_4, and that the susceptibility of magnetite is some 10,000 times greater than that of Fe_2O_3, the enormous error that can be introduced by traces of organic impurity will be realised. In fact, there is little doubt that the change in susceptibility brought about by heating pure Fe_2O_3 with any substance capable of reducing it would be a more delicate test for, say, organic carbon or hydrogen than any other at present known.

Table II: Magnetic properties of material in bars

Substance	H for μ_{max}	μ_{max}	k_v	Density	k_m	H_c	B_{rem}
Experiments with bars of compressed powders							
Fe_3O_4	207	3·50	0·198	1·73	0·115	49·2	248
Fe_2O_3 from oxidised Fe_3O_4	180	3·68	0·214	1·67	0·128	64·7	252
Lepidocrocite Easton	155	2·71	0·136	2·49	0·055	41·4	184
Lepidocrocite Siegen	180	2·47	0·117	2·27	0·0516	55	202
$MgO . Fe_2O_3$	233	1·83	0·066	1·80	0·0367	39	87
$CuO . Fe_2O_3$	155	5·37	0·348	3·02	0·115	54	324
Tilly Foster magnetite	103·5	4·39	0·27	2·84	0·095	25·9	114
Mineville magnetite	103	9·16	0·65	3·45	0·188	45	564
Experiments with solid bars cut from magnetite							
Tilly Foster	21	19·8	1·5	4·1	0·365	10·6	160
Mineville	55	35	2·7	5·12	0·527	20·7	1168

§ 5. COMPRESSED BARS

The modified Curie Balance previously described* (in which an electromagnet enables the field strength to be varied through very wide limits) was used for the determination of the susceptibilities of all the oxides in the form of powder, and so long as the value of the susceptibility was not high and the retained magnetisation not large it proved entirely satisfactory; but when used for comparing substances of such high permeability as magnetite and ferromagnetic Fe_2O_3 it was, on account of "end effect," unreliable, and tended to give too low values. Accordingly, the method was devised of moulding the powders into the form of rectangular bars 4 cm. long and 1 cm. by 1 cm. in a strong gunmetal mould, by mixing them into a stiff paste with some binding material (such as collodion, boiled starch, or a warm solution of photographic gelatine, which last was found on the whole the most satisfactory) and compressing them with a pressure of about 2 tons per square inch. After removal from the mould, which had removable sides held in position by bolts, they were allowed to dry in the air and finally introduced into the gap of a ring magnet, of which a full account has already been published†, where they formed an isthmus across the pole gap. Coils of fine wire wound on closely fitting formers of thin card or varnished paper were slipped over the specimens before insertion into the slots in the electromagnet. In this way the magnetic induction and permeability of the substances in the form of compact bars were obtained. Curves showing the relation between the permeability μ and the magnetising force H are given in the diagram, and experimental data are given in Table II. The results justified our expectation by giving much higher values than could be obtained from the loose powders, and enabled readings for the force H, at which maximum permeability μ occurred, to be recorded, and the apparent coercive

* *Phil. Trans. R. S.* A, **219**, Appendix (1919). † *Proc. Phys. Soc.* **31**, 299–315 (1919).

force* H_c and the retained magnetic induction B_{rem} to be measured in the usual way with a ring magnet. But we realise clearly that the values thus obtained are greatly influenced by the fact that the specimens still consist of particles, compressed, it is true, and held together with a binder, but not the solid material of which the powder consists. Comparison figures for bars moulded in this way out of powdered magnetite from both Tilly Foster and Mineville, N.Y. specimens with those obtained with bars cut from the solid material, are also given. The columns headed k_v, k_m show respectively the volume and mass-susceptibility in C.G.S. units.

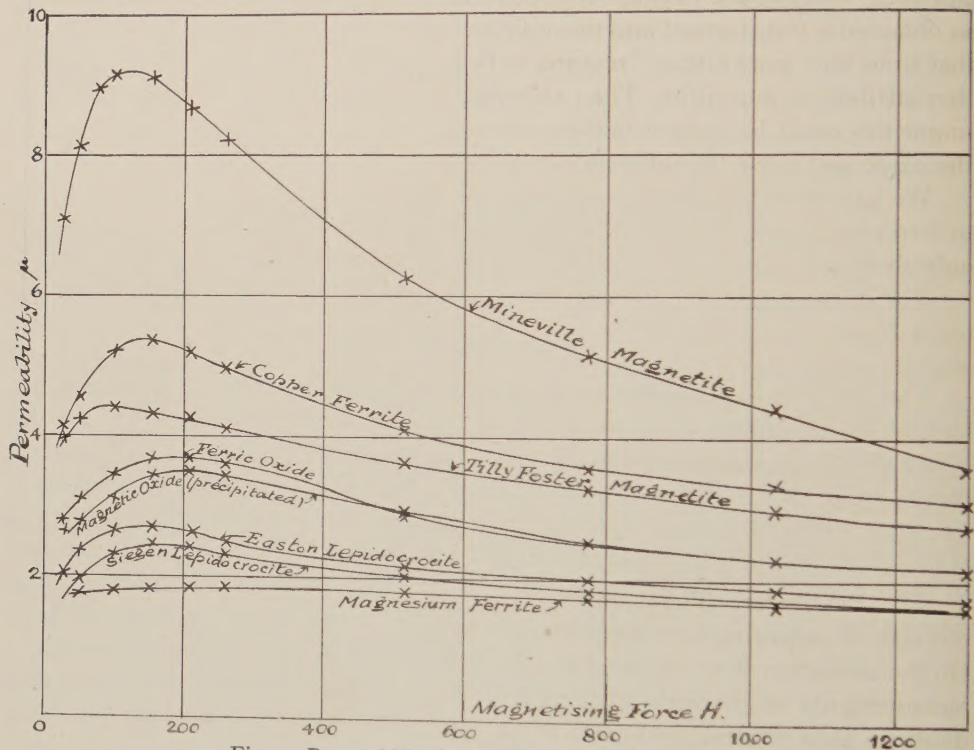

Fig. 1. Permeability curves of compressed powders.

It will be seen that results for the solid bars are between three and four times as great as for the bars of compressed powder, although the actual weight of dry binding material was only from 1 to 2 per cent. of the whole mass. The discontinuity in any such synthetic bar will naturally have a greater effect the higher the true permeability; hence it is believed that the values obtained in this way with the powders of lepidocrocite, artificial magnetic ferric oxide, etc. as detailed in Table II will be nearer their true values than with native magnetite. A comparison between powders and compressed bars has not been attempted on the grounds of the inequality in the corresponding magnetic forces employed.

* This coercive force is that particular reverse force which reduces the magnetic induction, B, to zero and not the force which when removed leaves the specimen in a demagnetised state.

NOTE ADDED NOVEMBER 23, 1928

J. Huggett and G. Chaudron in a recent communication* on magnetic sesquioxide of iron refer to it as having been discovered by Malaguti in 1863, whereas in fact the first recorded observation is that of Robbins four years earlier. They find a great difference in the thermal stability of the magnetic form according to whether it is prepared from commercial "pure ferric nitrate" or from the nitrate obtained by dissolving pure iron in nitric acid, recrystallising several times and then precipitating, washing and drying. After conversion into the magnetic form the product so obtained is transformed into the ordinary oxide by heating to 350°–450° C., while that from the "pure nitrate" requires to be heated to nearly 700° C. This difference they attribute to impurities. Their view may be correct, but it is difficult to see what impurities could be present in the commercial "pure nitrate" that could pass into the oxide and cause the difference in question.

We have found magnesia very potent in giving a high permanent susceptibility to ferric oxide, but even $MgO.Fe_2O_3$ containing about 20 per cent. of magnesia has only about a quarter of the susceptibility of ferromagnetic Fe_2O_3.

The curve given by Huggett and Chaudron shows the magnetisation of the oxide from the commercial nitrate to be higher on regaining room temperature after heating to 650° C. than it was initially.

They furnish no statement as to the magnitude of the magnetic field employed and the magnetisation is recorded as "deviation (cms.)": it is therefore impossible to obtain an approximate estimate of the susceptibility of the oxides with which they worked.

DISCUSSION

THE PRESIDENT: The subject of the magnetic susceptibility of crystalline minerals is acquiring new interest from studies of their crystal lattice structure. On the utilisation of compressed powders I would suggest that a series of magnetic measurements might with advantage be made on the same powder at various dilutions in a neutral medium such as starch and then extrapolation might be attempted to obtain the values for zero dilution.

AUTHORS' reply: In connection with the suggestion made by the President we would refer to experiments on mixtures of powdered magnetite and crushed quartz in varying proportions which have been described by Sosman and Hostetter. In these experiments they measured the pull in a non-uniform field†. Auerbach‡ used reduced metallic iron and wood powder; E. Wilson§ examined iron filings alone and mixed with ferromanganese by a ring method.

* *Comptes Rendus*, **186**, 1617–19 (1928).

† Sosman and Hostetter, *Trans. Amer. Inst. of Mining Engineering*, No. 126, June 1917; Benedicks, *Journal Iron and Steel Institute*, 1914, pp. 407–459.

‡ *Wiedemann's Annalen*, **11**, 353–394.

§ *The Electrician*, Oct. 5, 1900.

OLD SYNDICATE ROOM

DEMONSTRATIONS

Demonstration of a New Device for Thermostat Control, given by Mr H. F. T. Jarvis on October 28, 1928.

The device was designed to control the intermittent energisation of an immersion heater by which the temperature of a tank of water is kept constant. A large bulb full of toluol, immersed in the water, governs the level of a column of mercury on which floats a glass cylinder controlling two alternative wire-mercury contacts in circuit with solenoids which serve to rock in opposite directions a relay switch controlling the heating circuit. The relay switch comprises an evacuated bent glass tube containing a pool of mercury which connects a central wire terminal with one or other of two end terminals. By means of this arrangement a temperature constant within $\frac{1}{50}$° C. was maintained for a fortnight. The temperature can be adjusted by varying the quantity of mercury in the controlling column. A full description and diagram of the apparatus will be found in the *Journal of Scientific Instruments*, 4, no. 10 (July 1927).

Demonstration of Emulsions showing Chromatic Effects, given by R. H. Humphry, M.Sc., The Sir John Cass Technical Institute, on November 23, 1928.

The experiments shown were similar to those described by Holmes and Cameron*. The refractive index of glycerol (1·467) lies between that of benzene (1·502) and that of acetone (1·359). A stable emulsion of glycerol in acetone may be made by using pyroxylin (1 to 2 per cent.) as the emulsifying agent; such emulsions are milky, the glycerol being the disperse phase. By addition of benzene, which mixes with the acetone, the refractive index of the dispersion medium may be varied continuously and, when it is equal to that of the disperse phase, the emulsion becomes clear. Owing, however, to the large optical dispersion of the medium as compared with glycerol, equality of refractive index is only possible for light of a particular wave-length and, if the emulsion is viewed in white light, the transmitted light is of this particular region of the spectrum. Light of other wave-lengths suffers innumerable refractions and appears as scattered light of a colour complementary to that transmitted. The variation which is made by addition of benzene causes the transmitted light to change from violet, through the spectrum to orange, with corresponding change in the complementary colour. Subsequent addition of the other component (acetone) effects a colour change in the reverse direction. A series of emulsions exhibiting these chromatic effects was prepared and shown.

The effect of changing the temperature of such an emulsion was also demonstrated; a change of 1° C. causes a perceptible alteration of the colour, the change being, for a fall in temperature, in the same direction as for increasing proportion of benzene.

* *Journ. Amer. Chem. Soc.* **44**, 71 (1922).